Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week 09

Lecture 44: Wave in Plasma: Dispersion Relation

Hello dear students. So, we will continue our discussion on electron plasma wave. So, we have derived this set of equations earlier 1, 2 and 3 which are the governing equations and we have defined the perturbed variables as equation number 5. And the conditions which govern the plasma motion are given in equation number 6. Now, we have to substitute equation numbers 5 and 6 in equations 1, 2 and 3. The purpose is to linearize the governing equations.

So, we will take the continuity equation first. So, we can write dou by dou t the time derivative of n naught plus n1 plus del dot n1 plus n naught u1 plus u naught is equal to 0. So, the time derivative of this is going to be 0 and if you use the basic principle that second order terms in perturbation variables are going to be neglected. So, if you do that you can write the continuity equation dou n1 by dou t plus n naught del dot u1 is 0.

Similarly, we can use the perturbation for the momentum equation as well and we can write the momentum equation as Me n naught dou u1 by dou t is equal to minus e n naught e1 minus 3 k B T e n1 and del n1. And the Poisson equation as epsilon naught times del dot e1 is equal to minus e n. So, we call this equation as equation number 7, 8 and 9. So, these are the linearized governing equations in which we have only the perturbation variables like modified velocity, electric field and the number density. Now, the standard method is to assume sinusoidal variation of the perturbation variables.

$$\frac{\partial}{\partial t}(\cancel{n_0}^{\,0}+n_1) + \nabla\cdot(n_1+n_0)(u_1+u_0) = 0$$

$$\frac{\partial n_1}{\partial t} + n_0\,\bar{\nabla}\cdot\bar{u}_1 = 0 \qquad — (7)$$

$$m_e n_0 \frac{\partial \bar{u}_1}{\partial t} = -e n_0 \bar{E}_1 - 3 k_B T_e\, \bar{\nabla} n_1 \qquad — (8)$$

$$\epsilon_0 \bar{\nabla}\cdot\bar{E}_1 = -e n_1 \qquad — (9)$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i}$$

$$\frac{\partial}{\partial t}$$

$$(10)\quad \begin{cases} n_1 = n_0 \exp[i(kx-\omega t)] \\ E_1 = E_0 \exp[i(kx-\omega t)] \\ u_1 = u_0 \exp[i(kx-\omega t)] \end{cases}$$

Using (10) in (7) (8) & (9)

So, we will say that n1 the perturbed electron density is n naught exponential i kx minus omega t. So, you are describing a sinusoidal variation in one dimension or along the x direction or the electric field e1 as e naught exponential i times kx minus omega t and u1 the velocity as u naught exponential i kx minus. So, we know very well only the real part will have some physical meaning when you take these solutions. So, we can use these set of solutions, we call them as equation number 10 and we have to use them using set of equations in 10 in 7, 8 and 9. We will have to simply substitute them into this.

So, for example dou n1 by dou t would be n naught exponential i kx minus omega t times minus i omega. So, we take the derivatives as per this equations and the gradient or the del operator is going to be only i cap dou by dou x because we are considering a one dimensional picture. So, using this for the definition of del and the other type of derivative is the time derivative which is dou by dou t or you can as well choose d by dx directly because you are dealing only with one dimension. So, when you substitute these solutions into the linearized governing equations what you will get is minus i omega n1 plus n naught i k u1 is equal to 0 which is from the continuity equation. You see this the continuity equation, equation number 7 will give you this and we call this as equation number 11 or we can for simplicity or if you want to recall it easily let us say we call it as a and then i k epsilon naught e1 is equals to minus e n1.

$$(-i\omega)n_1 + n_0 i k u_1 = 0 \quad \text{— (a)}$$

$$i k \epsilon_0 E_1 = -e n_1 \quad \text{— (b)}$$

$$m_e n_0 u_1 (-i\omega) = -e n_0 E_1 - 3 k_B T_e i k n_1 \quad \text{— (c)}$$

ω vs k

$$\text{(b)} \quad E_1 = \frac{-e n_1}{i k \epsilon_0}$$

$$m_e n_0 u_1 (-i\omega) = -e n_0 \left(\frac{-e n_1}{i k \epsilon_0} \right) - 3 k_B T_e i k n_1$$

$$-i\omega m_e n_0 u_1 = \left[\frac{e^2 n_0}{i k \epsilon_0} - 3 k_B T_e i k \right] n_1$$

$$\text{(a)} \quad n_1 = \frac{-n_0 i k u_1}{-i\omega} = \frac{n_0 k u_1}{\omega}$$

This is equation number b and this is resulting from the Poisson equation and the momentum equation when linearized and upon substitution of the sinusoidal solutions will be u1 minus i omega is equals to minus e n naught e1 minus 3 k B T e i k n1. So, we have a, b and c. Now, we have to eliminate some variables and obtain the dispersion relation which is between omega versus k. So, if there is a wave in this picture then the ratio of omega by k will give you the phase velocity and the ratio of d omega by dk will give you the group velocity. From equation number b, we can write e1 as equal to minus e n1 divided by i k epsilon naught.

Substituting this in equation number c, we will get m e n naught e1 minus omega is equals to minus e n times minus e n1 divided by i k epsilon naught minus 3 k B T e i k n1. So, which will be minus i omega m e n naught u1 is equals to e square n naught by i k epsilon naught minus 3 k B T e i k times n1. Now, from equation number a, we can write n1 is equals to minus n naught i k u1 divided by minus i omega which is equals to n naught k u1 by omega. So, we can use this n1 here. So, we have n1 written as n naught k u1 by omega.

$$m_e n_0 \cancel{u_1}(-i\omega) = \left[\frac{e^2 n_0}{ik\epsilon_0} - 3k_BT_e\, ik\right]\left[\frac{n_0 \cancel{i}k\cancel{u_1}}{\cancel{i}\omega}\right]$$

$$\frac{1}{2}mv^2 = \overset{T.E}{\overline{k_BT}}$$

$$m_e n_0(-i\omega) = \left[\frac{e^2 n_0^2 k}{ik\epsilon_0 \underline{\omega}} - \frac{3k_BT_e\, ik^2 n_0}{\underline{\omega}}\right]$$

$$\frac{k_BT}{m} = \frac{v_{th}^2}{2} \Leftarrow v^2 = \frac{2k_BT}{m}$$

$$v_{th} = \sqrt{\frac{2k_BT}{m}}$$

$$-i\omega^2 = \left[\frac{e^2 n_0^2 k}{ik\epsilon_0(m_e n_0)} - \frac{3k_BT_e\, ik^2 n_0}{m_e n_0}\right]$$

$$\omega_p = \sqrt{\frac{n_0 e^2}{m_e\epsilon_0}}$$

$$\omega^2 = \left[\frac{e^2 n_0^{\cancel{2}}\cancel{k}}{-i^2\cancel{k}\epsilon_0(m_e\cancel{n_0})} - \frac{3k_BT_e\cancel{i}k^2\cancel{n_0}}{-\cancel{i}\, m_e\cancel{n_0}}\right]$$

$$\boxed{\omega^2 = \frac{e^2 n_0}{m_e\epsilon_0} + \frac{3k_BT_e k^2}{m_e}}$$

We can use n1 into this expression as this. Now, doing it, what we have is a simple algebra. I am doing it just so that we have continuity and you can understand it better. This i omega is equals to e square n naught divided by i k epsilon naught minus 3 k B T e i k times n naught i k u1 divided by i omega. We can cancel u1 which appears on both sides and we can write m e n naught minus i omega is equals to e square n naught square k divided by i k epsilon naught omega minus 3 k B T e i k square n naught divided by omega.

So, we will take the omega on the denominator to the left hand side and we will write minus i omega square is equals to e square n naught square k divided by i k epsilon naught times m e n naught, n naught is unperturbed electron density minus 3 k B T e i k square naught divided by m e n naught. So, omega square now becomes e square n naught square k divided by i k minus i square k epsilon naught times m e n naught minus 3 k B T e i k square n naught divided by minus i m e n naught. So, we will have this n naught, all these things will be cancelled. So, omega square can now be written as e square n naught by m e epsilon naught plus 3 k B T e this minus minus becomes plus k B T e k square divided by m e. This is an expression that we got for omega in terms of k, the propagation constant.

$$\omega^2 = \frac{e^2 n_0}{m_e \epsilon_0} + \frac{3 k_B T_e k^2}{m_e}$$

$$\frac{v_{th}^2}{2} = \frac{k_B T_e}{m_e}$$

$$\boxed{\omega^2 = \omega_p^2 + \frac{3}{2} v_{th}^2 k^2} \quad — \quad (1)$$

Plasma frequency

Non-zero thermal velocity of e^-

$$v_g = \frac{d\omega}{dk} \neq 0$$

$\rightarrow$ Wave propagation exists

$$\omega_{pe} = \sqrt{\frac{ne^2}{m_e \epsilon_0}}$$

$$v_{th} = \sqrt{\frac{2 k_B T_e}{m_e}}$$

$$\lambda_D^2 = \sqrt{\frac{\epsilon_0 k_B T_e}{n_0 e^2}}$$

So, omega square is equals to e square n naught by m e epsilon naught plus 3 k B T e k square by m e. Now, if you consider the electron to have thermal energy or temperature non-zero temperature, you can write half m e square is equals to k B T. So, what is this? This is the thermal energy. So, this is the kinetic energy. So, you can write V square is 2 k B T by m or we call V thermal as square root of 2 k B T by m.

So, this k B T by m can be written as V square by 2. k B T by m is V thermal energy square by 2. And this parameter e square n naught by m e epsilon naught is the plasma frequency. We call it as omega p which is equals to square root of n naught e square by m e epsilon naught. So, we can substitute all of these things into the equation and we can rewrite as omega square is equals to e square n naught by m e epsilon naught plus 3 k B T e times k square the wave vector by m e.

So, omega square has to be written as omega p square plus so V T H square by 2 is k B T by m, k B T e by m or m e. So, k B T this can be replaced this part k B T e by m e as V T H square. So, it can be written as 3 by 2 V T H square and k square. So, this is the expression. This is the most important expression because it gives us a lot of information about the plasma wave.

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{th}^2$$

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} \boxed{\frac{k_B T_e}{m_e} \frac{\epsilon_0}{\epsilon_0}} \frac{n_0 e^2}{\boxed{n_0 e^2}} k^2$$

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2\, 2\lambda_D^2 \underbrace{\frac{n_0 e^2}{m_e \epsilon_0}}$$

$$\omega^2 = \omega_{pe}^2 + 3k^2 \lambda_D^2 \omega_{pe}^2$$

$$\boxed{\omega^2 = \omega_{pe}^2 \left(1 + 3k^2 \lambda_D^2\right)}$$

Plasma oscillations $\Rightarrow v_g = 0$

e^- plasma wave $(T_e \neq 0)$ $\Rightarrow v_g \neq 0$

So, what is omega? Omega is the angle of frequency of the wave passing through the plasma due to the electron oscillations. So, earlier we have realized electron oscillations do not propagate. They oscillate and these oscillations do not constitute a wave or a wave train. What is a wave? A wave is something which carries information or energy from one point to another point. So, the plasma oscillations do not constitute a wave as long as you consider the plasma to be cold.

But if you impose a condition that plasma particles or the electrons if they have some thermal energy, these electrons by the virtue of the thermal energy will be transported from one layer to another layer. When they are being transported, you are actually conveying the message that they are actually taking some information from one layer to another layer of plasma. So, another layer of plasma knows or the information is there about an earlier layer or a layer before that. So, there is an information passage from one point to another point. So, by considering the electron temperature to be non-zero, we have obtained a different expression for omega and this expression seems to be a function of k.

That means, Vg is when you define Vg d omega by dk is not equal to 0. What does it mean? This simply means always just remember that wave propagation exists. Let us look at the properties of this wave. So, omega square is the angular frequency of the wave is plasma frequency plus something. So, this is the plasma frequency.

This is the plasma frequency which is a characteristic of the plasma oscillation and this part is because of the non-zero thermal velocity of electron. Now let us try to rewrite this expression in terms of something that we know. So we can define, we know very well that omega pe is square root of n e square by Me epsilon naught. Vth is 2 kB Te by Me and if we recall the Debye s length lambda d square as epsilon naught kB Te by n naught e square. We can write this as equation number 1 in terms of lambda d.

How do we do it? So, omega we consider the original expression omega square is omega pe or electron plasma frequency plus 3 by 2 k square Vth square. So, we multiply and divide with something just to facilitate the inclusion of Debye s length into this expression. So, 3 by 2 kB Te by Me times epsilon naught divided by epsilon naught times n naught e square by n naught e square. So, this one kB Te is a measure of the Debye s length. So, we can write omega square is omega pe square plus 3 by 2 into k square.

$$\boxed{v_p = \frac{\omega}{k} = \sqrt{\frac{3}{2}}\, v_{th} \frac{\omega}{\sqrt{\omega^2 - \omega_{pe}^2}}}$$

$$v_p = \frac{\sqrt{\omega_p^2 + \frac{3}{2} k^2 v_{th}^2}}{k}$$

Phase velocity

$$v_g = \frac{d\omega}{dk} = \frac{3}{2} \frac{k}{\omega} v_{th}^2$$

$$\boxed{v_g = \frac{3}{2} \frac{v_{th}^2}{v_p}}$$

So, if I would write k square here to k square 2 lambda d square into n naught e square by Me epsilon naught. So, omega square is omega pe square plus 3 k square lambda d square omega pe square. So, this is again the plasma frequency. How did we get this term? So, in order to accommodate lambda d, we have to multiply with epsilon naught divided by epsilon naught multiplied with n naught e square divided with n naught e square. Just so that we can call this kB Te epsilon naught by n naught e square as lambda d square and the remaining terms seems to be resembling the plasma frequency.

So, the dispersion relation can now be written as omega square is omega pe square times 1 plus 3 k square lambda d square. This is another version of the dispersion relation of the electron plasma wave. So, this wave is called as the electron plasma wave. So, we have started from plasma oscillations which gave a situation Vg is equal to 0. The electron plasma wave by assuming the temperature of electron not to be equal to 0 gave Vg as not equal to 0.

That means we can now call these oscillations to be responsible for a wave activity. Now, the task is to find out the phase velocity and group velocity for which we can use this basic definition. So, we can write Vp is simply omega by k. We can express it as Vth times omega divided by omega square minus omega pe square. So, it is basically omega is square root of omega pe square plus 3 by 2 k square Vth square divided by k.

This is your Vp. So, I have rearranged the terms so that Vp can be written like this. What is this? This is the phase velocity. So, this velocity tells you how a given phase in the wave is propagating or how it is moving. Now, when you calculate Vg, the group velocity which is d omega by dk which is equal to 3 by 2 k by omega Vth square. d omega by dk is just take a derivative, you will get this or Vg can be simplified to write it as 3 by 2 Vth square divided by Vp.

$$\boxed{\omega^2 = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2}$$

The group velocity is 3 by 2 times the thermal velocity square divided by Vp. So, this is something about the electron plasma wave which is actually transporting energy from one point to another point. Now, if you want to understand more features about this dispersion relation, like I said before dispersion relation tells you all the required information about the plasma wave or any wave. So, we have omega square is equal to omega p square or omega p square plus 3 by 2 k square Vth square. So, in the next class, we will try to understand what is the graphical representation of this dispersion relation and what kind of emphasis can be drawn from the graphical representation. Thank you.