Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week 09

Lecture 43: Wave in Plasma: Plasma Oscillation

Hello dear students. In continuation to our last lecture on plasma oscillations, we have learned what is perturbation theory and how it is useful to understand waves which are of very small amplitudes. So, this is the plasma oscillations. So, using the perturbation theory in combination with the governing equations, we can derive equations which are minus i M e omega u1 is equals to minus e e1 minus i omega n1 plus n naught i k u1 is equals to 0 and epsilon naught i k e1 is equals to minus e n1. So, we have obtained let us say we call this equation as A, B and C. Just to recall this equation, equation number A was obtained from the momentum equation.

The second equation was obtained by using the perturbation variables into the continuity equation. In the third one, the equation E was from the Poisson equation. Now, the task is to couple these equations and try to understand how we get the dispersion relation. The objective is to obtain the dispersion relation.

What is dispersion relation? It is an expression between omega versus k, propagation vector k and angular frequency omega. So, we can eliminate few variables, the perturbation variables from this to obtain this expression. Perfusion relation will straight away give you the formula for velocity of the wave which is propagating inside the plasma. So, why this particular set of equations will represent anything related to the plasma? We have considered the governing equation such that all physical variables which are influenced by the propagation of waves are included in the governing equations. So, let us say we can use equation number C.

Plasma Oscillations

$$-im_e\omega u_1 = -eE_1 \quad \text{— (a)} \qquad M.E$$

$$-i\omega n_1 + n_0 iku_1 = 0 \quad \text{— (b)} \qquad C.E$$

$$\epsilon_0 ikE_1 = -en_1 \quad \text{— (c)} \qquad \underline{P.E}$$

Dispersion relation (ω vs k) $= \underline{0}$

$$(c) \Rightarrow E_1 = \frac{-en_1}{\epsilon_0 ik}$$

$$(b) \Rightarrow n_1 = \frac{n_0 iku_1}{i\omega} = \frac{n_0 ku_1}{\omega}$$

$$E_1 = \frac{-e}{\epsilon_0 i\cancel{k}} \times \frac{n_0 \cancel{k} u_1}{\omega}$$

$$\boxed{E_1 = \frac{-en_0u_1}{\epsilon_0 i\omega}}$$

Using C, we can write E1, the perturbed electric field as minus En1 divided by epsilon0 Ik. And from equation number B, we can write En1 is equals to N0 Ik U1 divided by i omega which is equals to N0 k U1 by omega. So, using this value of N1 into E1, we can write E1 as minus E divided by epsilon0 Ik times N1 which is N0 k U1 by omega. So, E1 is thus minus En0 U1 by epsilon0 i omega. So, we will use E1 further in our calculations.

Substituting E_1 in (a)

$$\frac{\omega}{K} = v$$

$$-i m_e \omega u_1 = -e\left(\frac{-e n_0 u_1}{\epsilon_0 i \omega}\right)$$

$$-\cancel{i} m_e \omega \cancel{u_1} = -\cancel{i}\, \frac{n_0 e^2 \cancel{u_1}}{\epsilon_0 \omega}$$

$$m_e \omega = \frac{e^2 n_0}{\epsilon_0 \omega}$$

$$\omega^2 = \frac{n_0 e^2}{m_e \epsilon_0}$$

$$\boxed{\omega = \sqrt{\frac{n e^2}{m_e \epsilon_0}}}$$

Plasma frequency

ω vs k

- Cold = $\boxed{T_e = 0}$
- Collisionless
- $B = 0$
- Infinitely Extended
- Ions are in the background

So, we can substitute E1 in expression A. Substituting what E1 in A. What do we have is minus i Me omega U1 is equals to minus E times minus En0 U1 divided by epsilon0 i omega. Using some simple algebraic simplifications, we can write i Me omega U1 is equals to minus i times N0 e square U1 by epsilon0 omega. E1 gets cancelled, we can write Me omega is equals to e square N0 by epsilon0 omega.

So, omega square is N0 e square by Me epsilon0 or omega is square root of Ne square by Me epsilon0. So, we wanted to achieve the dispersion relation between omega versus k. But what you have got is an expression for omega which is not dependent on the propagation vector k. But this expression is very familiar to us. This expression is nothing but angular frequency of plasma oscillations or generally we call it as the plasma frequency.

$v_p = \frac{\omega}{k}$; $v_g = \frac{d\omega}{dk}$

Plasma Oscillations do not propagate as waves

$T_e \neq 0$

$e^- \longrightarrow \|$

— warm ($T_e \neq 0$)

— Collisionless

Now, if we try to understand the physics of this, we have started with a simple picture where plasma is considered to be collisionless. The basic conditions of plasma are the plasma is cold, the plasma is collisionless and the magnetic field is zero and plasma is infinitely extended. These are the basic conditions with which we started understanding the plasma oscillations. Let us try to see how these basic conditions are related to the dispersion relation or plasma frequency expression that we have obtained at the end. When you say plasma is cold, you are assuming or you are imposing the fact that the temperature of the electron is zero.

So, there is no energy associated with the electron which will make it move from one point to another point or there is no thermal energy available to the electron. So when you say that, so electron is completely dependent on the electric field or electrons motion is completely guided by the electric field and collisionless. So, there are no collisions in the plasma. There is no momentum transfer that is happening between the electrons and the ions. And in addition to all this, we also assume that ions are in the background.

Governing equations $\quad T_e \neq 0$, Adiabatic

Continuity equation $\quad \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e) = 0$ — ①

Momentum equation $\quad m_e n_e \left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e \cdot \nabla)\vec{u}_e \right] = -e n_e \vec{E} - 3 n_e k_B T_e \vec{\nabla} n_e$ — ②

Poisson equation $\quad \epsilon_0 \left[\vec{\nabla} \cdot \vec{E} \right] = e(n_i - n_e)$ — ③

$$P = c\rho^{\gamma}$$

$$\frac{\nabla P}{P} = \frac{c\gamma \rho^{\gamma-1} \nabla \rho}{c\rho^{\gamma}} = \gamma \rho^{-1} \nabla \rho$$

$$= \gamma \frac{\nabla \rho}{\rho} = \frac{\gamma \nabla n_e}{n_e}$$

$$P = n_e k_B T_e$$

$$\nabla P = \gamma P \frac{\nabla n_e}{n_e} \Longrightarrow \nabla P = \gamma n_e k_B T_e \frac{\nabla n_e}{n_e}$$

So, we constructed a picture in which the ions are at rest because of the large mass and they are not moving anywhere. It is the electrons which are trying to oscillate and we have considered the magnetic field to be zero. There are no magnetic field perturbations on to the governing equation or we have simply made the Lorentz force term, the v cross B term in the momentum equation as zero. And plasma is infinitely extended. So, what it means is that you are considering a picture in which you have displaced by applying an external force some electrons from its original position and immediately when you do that electric fields would be set up and this electric field will always try to bring back the electrons which were removed.

And when the electrons are coming back, they will not simply come back and sit in their original position rather it will create some oscillations in which electrons will oscillate. So, these oscillations are generally referred to as plasma oscillations and we have derived the same expression for plasma frequency using a different approach earlier. But this approach which is basically out of the MHD equations also leads to the same expression. The most important fact that we have to understand about this is that omega the angular frequency seems to be dependent on the number of electrons, the charge of electron, mass of electron and it is not dependent on k. So, if you say omega by k is the velocity.

So, this expression tells you that the angular frequency is not a function of k or these oscillations are the ones in which energy is not transported as a wave into the plasma. So, the oscillations are local. These oscillations does not constitute a plasma wave. So, you cannot expect a wave motion. What is a wave? Wave is a transport of energy from one point to another point in the medium.

Let us say in plasma medium in this case. Wave can also travel without a medium that is a different story. But in this picture, the plasma wave is supposed to be something which is transporting information or energy from one point to another point. So, this says the velocity, the Vp if you talk about the velocities, Vp is omega by k and Vg is d omega by dk. So, we can say that in this picture, there is no energy transported or the oscillations can be of just around the mean position and the oscillation frequency can be anything.

$$\boxed{\nabla P = \gamma k_B T_e \vec{\nabla} n_e} \quad — (4)$$

$$\gamma = 1 + \frac{2}{f} = (f=1) \Rightarrow \boxed{\gamma = 3}$$

$$\boxed{\nabla p = 3 k_B T_e \vec{\nabla} n_e} \quad — (4)$$

we have to solve $\underline{(1)(2)(3)}$ (Linearizing)

$$\left.\begin{aligned} n_e &= n_0 + n_1 \\ u_e &= u_0 + u_1 \\ E_e &= E_0 + E_1 \end{aligned}\right\} (5)$$

The phase velocity of such oscillations can be anything. It can be even greater than the velocity of light, but there is no energy transport. What could be the reason which is making the plasma oscillations not constituting a wave motion is the most important point that we have to understand. What is the cause? The plasma oscillations do not propagate as waves. The major conclusion is that we can write it down.

The plasma oscillations do not propagate as waves. The most important conclusion is that the plasma oscillations do not propagate as waves. What is the cause? Let us say if the plasma constituents, the electrons have a thermal velocity, then they can travel into the adjacent layers of plasma and they can modulate the environment in those adjacent layers by which they are simply making the wave or they are simply transferring information from one layer to another layer. Because you have imposed the fact that the temperature of electron is zero, if you make it non-zero, then there is a possibility that the electron because of its thermal energy that it poses by the virtue of its temperature, it can get transported into the adjacent layers of plasma and it can propagate the information from one point to another point. This may be the main reason that our picture of plasma oscillations at this point of time is devoid of any wave motion.

Uniform, neutral & collisionless plasma

$$\left.E_0 = 0 \ ; \ u_0 = 0 \ , \ \frac{\partial n_0}{\partial t} = \frac{\partial u_0}{\partial t} = \frac{\partial E_0}{\partial t} = 0 \right\} \ (6)$$

(5) & (6) in (1) (2) (3)

So then what do we do? When we have thermal motion, we can say that there are pressure gradients which will automatically build up and in addition to the electromagnetic term or the Lorentz force term, we will also need to accommodate the pressure gradient force which will be a natural consequence of electron having a temperature. So let us say we modify the picture slightly and in comparison to the original picture which is the plasma is cold, collisionless, the magnetic field is zero and plasma is infinitely extended and ions because of their heavier mass are constituting the background of these oscillations. So we will consider a picture in which the plasma is warm, which means the electron temperature is non-zero and at the same time the plasma is still collisionless because collisions can complicate the entire mathematical description in a very different way but we will take it up at the later point of time. But for now let us try to see what is the possibility which can accommodate wave motion from one point to another point within the plasma either the information is going to be carried out from one point to another point by the electrons, what is the consequence? So now let us consider the electrons to be moving, the electron fluid equations can be rewritten as per this choice. So we have to modify the set of governing equations that we have been using in order to understand the motion of plasma inside.

So we will still consider the magnetic field to be zero. So let us say the governing equations which you require, the set of governing equations are so we will first take the electron temperature to be non-zero and we will assume that the picture is adiabatic in nature. What does it mean? There is no exchange of temperature or energy from the surroundings. So the first equation is dou NE by dou T plus del dot NE ue is equal to zero. What is it? This is the continuity equation.

What is it telling? at which the electron density is changing with respect to time is related to the divergence of this mass flux, the NE ue term. Let us say we call it as equation number one. The second one is ME NE mass of electron times the number of electrons per unit volume which is this is equal to the density itself times the total derivative dou by dou T plus ue dot del times ue is equal to minus E minus del P. What is the change here? The change is you have included the pressure gradient as an additional term on the right hand side of the momentum equation. Then, we have the displacement of electrons will lead to the creation of localized electric fields.

del dot E is E times NI minus NE. Let us say we call it as 3 and this equation is the Poisson's equation. Now we have to solve these three equations and try to see if the omega is still independent of k or not. The first question that we should ask ourselves is that the change is only the electron temperature not equal to 0. Is there any provision that the temperature is directly written or available in these variables which are there in the three equations? No, but this seems to be the change that we have brought in.

Let us see how the temperature is actually responsible for setting up the pressure gradients in the plasma. If you bring the ideal gas equation P is C times rho power gamma where P is the pressure C is a constant rho is density gamma is the ratio of specific heats. We can take a derivative del P over P dividing with the same expression we have C times gamma rho power gamma minus 1 divided by C times rho power gamma which is equals to gamma times rho power minus 1 del rho which is also equals to gamma times del rho by rho. So, I can assume that the change in the density is equivalently reflected in change in the number of electrons per unit volume or the number density del N e by N e. So, this is directly delta P by P is equals to gamma times delta N e by N e.

But where is the temperature is still not part of this equations. So, we can write P is equals to N k B T we know that P is equals to N k B T e. So, we can write del P as gamma N e k B T e delta N e by N e. So, we have delta P is equals to gamma P delta N e by N e. So, this is for the P that you see on the hand side here I am substituting N k B T e N e N e k B T e times.

So, as a result cancelling N e we have delta P is equals to gamma k B T e del N e. So, this is let us say we call this equation as 4. Now what you have to understand is by making the electron temperature non zero we have got delta P. Otherwise in our earlier picture since we have made an approximation that we are considering cold plasma the electron temperature was considered to be zero and thus the equation that we dealt with contained only this term the electromagnetic force term on the right hand side. Now let us say we are considering an one dimensional picture in which gamma is to be written as 1 plus 2 by F.

What is F? F is the number of degrees of freedom and we say that only one degree of freedom is available for this picture. So, F is 1. So, we will get gamma as 3. So, we can write delta P as 3 k B T e delta N e. This we are going to use in the we will call the same equation 4 equation 4 just with the help of gamma being 3.

So, now we have to understand how we can solve this 1, 2, 3 and 4. 4 is actually a part of 2. So, we can now write delta P or minus 3 N e k B T e and delta N e. Now let us say this is equation number 2. So, we have to now use we have to solve equation 1, 2 and 3 or we have to start by linearizing the 3 equations.

Now since we have made the set of equations, the variables which are going to be influenced or affected or perturbed are these. So, the number of electrons can be written as N naught plus N 1. The velocity of electrons is U naught plus U 1. The electric field is E naught plus U 1. So again just for the sake of recalling, these are the perturbed these are the equilibrium parts and a variation of these with respect to time is always 0 and these are the perturbed parts which actually account for the perturbation or the for the variance away from the mean values.

And any wave activity has to be described or will be described in these variables. Now what do we do? We make one more assumption. So we have considered a uniform, neutral and collisionless plasma. This is what we have considered. Uniform as in wherever you go, you will find particles of equal numbers.

Neutral as in the total number of electrons is equal to the total number of ions. Collisionless as in there are no collisions between the particles, electrons or the ions. So which means we will get when it is uniform, E naught is equal to 0 and U naught is also 0 because it is at rest and dou N naught by dou T is equal to dou U naught by dou T is equal to dou E naught by dou T is 0. So let us say we call the set of perturbations as this set of perturbations as equation number 5 and this as equation number 6. So now we have to use equation number 5 and 6 into the governing equations 1, 2, 3.

So we will continue in the next class. Thank you.