Plasma Physics and Applications

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Week 09

Lecture 42: Wave in Plasma: Perturbation Theory

Hello dear students. We are in the process of understanding wave motion in plasma. This is waves in plasma. In the last lecture, we have seen the plasma is very rich in waves, plasma is very rich in wave phenomena. And we have seen what is the phase velocity Vp is omega by k and Vg is d omega by dk. The phase velocity tells about individual waves and the group velocity tells about wave packet of wave or a group of wave.

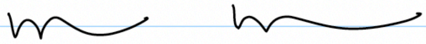
So here the most important thing that you have to understand is when the group velocity becomes 0, we can say that no information is transmitted. The wave is not carrying any information into the space. Now what we are going to do in this lecture is we are going to see how plasma allows waves to propagate through it and if a wave is propagating through it, we have to be able to address few important characteristic properties of the waves. So how do we do it? We will use the perturbation theory or linear perturbation method to obtain the entire task of this lesson is to obtain what is called as a dispersion relation.

What is a dispersion relation? Dispersion relation is simply an expression or between omega and k. The wave vector, the propagation constant and the angular frequency. So this is the main task. So for different types of waves, the nature of this dispersion relation will be different. So if you know the dispersion relation, you can establish all the different properties about the wave.

Waves in Plasma

— Plasma is very rich in wave phenomena.

— $V_p = \frac{\omega}{K}$; $V_g = \frac{d\omega}{dk}$



Individual waves   Packet of wave

$V_g = 0 \Rightarrow$ No information is transmitted

— Perturbation Method

— Dispersion relation

$(\omega / \& K)$

— Waves of small amplitude

You can calculate the velocity and all other how the wave propagates, how it dissipates, how it travels in a medium, all of it, how momentum is transferred by this wave. So what kind of waves do we want to study in plasma? Basically we are going to confine ourselves to the waves which are waves of small amplitude. Why? So that we can use the linear perturbation theory. And how do we use the linear perturbation theory? We will figure out what are the variables which can be influenced by the passage of these waves. We will perturb those variables, substitute them into the governing equations, solve the simultaneous equations and then obtain a dispersion relation.

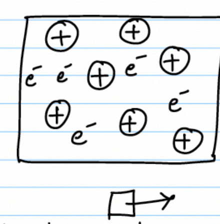
Wave motion in Plasma

— Plasma frequency.

— Ions do not move

— $\bar{B} = 0 \Rightarrow \bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t} = 0$

— $\vec{E} = E\hat{i}$



— Ions contribute to a uniform background.

— Plasma is homogeneous & Infinitely Extended

— cold $\Rightarrow$ No random motion of constituent particles

$\bar{E} = -\bar{\nabla}\varphi$, $\bar{\nabla} \times \bar{E} = 0$

This is the basic procedure that we generally follow when we are using the linear perturbation theory. And perturbation theory also has a limitation that the perturbed variables are always very small in comparison to the regular or the background variables. Let us consider a simple case of wave motion in plasma. So fundamental tendency of the plasma is to establish charge neutrality. So if you consider a plasma to be made up of electrons and ions and we know that the ions are heavier and given any temperature, it is more reasonable to understand that electrons will pick up this kinetic energy easily and they will move very fast in comparison to the ions.

1) $$m_e n_e \left[\frac{\partial \bar{u}_e}{\partial t} + (\bar{u}_e \cdot \bar{\nabla})\bar{u}_e\right] = -e n_e \bar{E} \quad \text{— (1)}$$

C.E 2) $$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e u_e) = 0 \quad \text{— (2)}$$

Poisson eq 3) $$\epsilon_0 (\bar{\nabla} \cdot \bar{E}) = e(n_i - n_e) \quad \text{— (3)}$$

$\Rightarrow$ Dispersion Relation

ω vs k

(I)
$$n_e = n_0 + n_1$$
$$u_e = u_0 + u_1$$
$$E = E_0 + E_1$$

$\Rightarrow$ Perturbed quantities due to oscillations

(n_0, u_0, E_0:) Equilibrium quantities
— Basic state
— Unperturbed State

$n_1 \ll n_0$

So let us say the ions are stationary in our picture and ions are not moving anywhere. So they constitute, if this is your plasma system, you consider all the ions to be stationary and they are not moving for example. They constitute a constant positive charge background and it is the electrons which are moving from place to place. Now we are trying to establish the nature of waves, how something will oscillate, how it will create a disturbance. Now if you consider these electrons to be moving and by applying an external force or an influence, if you are able to displace some of these electrons away from their position, then we know very well that the plasma has a tendency to keep itself neutral.

$$\nabla n = 0 \; ; \; u_0 = 0 \; \& \; \bar{E}_0 = 0$$

$$\frac{\partial n}{\partial t} = \frac{\partial u_0}{\partial t} = \frac{\partial \bar{E}_0}{\partial t} = 0 \quad \text{— (a)}$$

Using (I) & (a) in (1) & (2) & (3)

$$m_e n_e \frac{\partial \bar{u}_1}{\partial t} = -e \bar{E}_1 \quad \text{— (4)}$$

$$\frac{\partial n_1}{\partial t} + n_0 \vec{\nabla} \cdot \vec{v} = 0 \quad \text{— (5)}$$

$$\epsilon_0 (\bar{\nabla} \cdot \bar{E}_1) = -e \left(n_i - (n_0 + n_1) \right)$$

$$\Longrightarrow \epsilon_0 (\vec{\nabla} \cdot \vec{E}_1) = -e n_1 \quad \text{— (6)}$$

Perturbed variables are varying sinusoidally

$$u_1 = u_1 \exp[i(kx - \omega t)] \hat{i}$$

It will shield the electric fields applied from the outside. So if you displace some electrons, then what will happen is by the means of displacing, you are setting up an electric field and the plasma will adjust itself by moving such that the electric field that is created additionally is nullified. So in this process what happens, the electrons will move to the point where the electric field is set up and if the electrons are moving, they have this mass and they have some inertia. So generally what happens is they cannot simply come and nullify whatever the positive charge region that is created. So as a result, they will always overshoot and in the process of neutralizing the external electric field, the particles will oscillate.

So this oscillation can be thought of as a wave motion. So they will always oscillate about the equilibrium position and this characteristic frequency of oscillation is we also know it very well as the plasma frequency. Now what we will do in today's class is that we will obtain an expression for plasma frequency not by the old method, we will use the perturbation method and the MHD equations to obtain or to characterize the plasma frequency. So for this picture, we assume that the ions remain fixed. We will assume that the ions say they do not move, they remain fixed to their positions and we will use the linear perturbation theory.

$$\left.\begin{aligned} n_1 &= n_1 \exp[i(kx - \omega t)] \\ E_1 &= E_1 \exp[i(kx - \omega t)]\, \hat{i} \end{aligned}\right\} \quad (b)$$

Substituting (b) into (4), (5) & (6)

$$-i m_e \omega u_1 = -eE_1 \quad \text{--- (7)}$$

$$-i\omega n_1 + n_0 i k u_1 = 0 \quad \text{--- (8)}$$

$$\epsilon_0 i k E_1 = -e n_1 \quad \text{--- (9)}$$

So in addition to this, we have to make some more assumptions. Let us say we are considering a picture in which the magnetic field is 0 or there is no magnetic field and as a consequence, we can say that del cross E which is equals to minus dou B by dou t is also equal to 0. And we assume the electric field E to be along the positive x direction. And the ions because of the heavier mass are at rest and ions contribute to a uniform background. And we take that plasma is homogeneous and infinitely extended.

And finally, we assume that the plasma is cold which means there are no random motions of the constituents. In such a situation, if you create an electric field by displacing electrons physically or by applying an external force, what will happen? The resulting oscillations are known as plasma oscillations and these oscillations are a way by which plasma tries to establish charge neutrality within itself. So we can say that using it, so we have E is equals to minus del phi, we know this and the electric field to be along the x direction and we can write del cross E is equals to 0. Now, we will bring in the set of governing equations that we need for solving this problem. Let us say the first equation that we need is the momentum equation which is for the electron M e N e dou u e by dou t plus u e dot del times u e is equals to minus e into N e e.

So we know all of these parameters. I have considered only the momentum equation for an electron because I am assuming that ions are at rest and they do not contribute to anything. This is the first momentum equation. Then we need the continuity equation which is dou N e by dou t plus del dot N e u e is equals to 0. Let us say we call this as 1, we call this equation as 2 and then we need the Poisson equation which says epsilon 0 times del dot E should be equal to the charge density.

We write the charge density as E times Ni minus N e. If the plasma is completely neutral, there is no reason for us to assume that the number of electrons will not be equal to the number of ions. But if you have created a displacement and there may be a region where the number of electrons are more in comparison to the number of ions or the other way. Now these are the three governing equations that we need to obtain the dispersion relation for the plasma waves or oscillations which will be developed when collection of electrons is displaced from one point inside the plasma. Now what we have to do? We have to work out with these equations.

The task is to obtain the dispersion relation. What is dispersion relation? Dispersion relation is a relation between omega versus k. Now here we use the perturbation method. What is the perturbation method? We take what are the variables which may be influenced. Let us say for example, Ne the number of electrons in this picture the number of electrons can get perturbed.

Number of electrons is equal to N0 plus N1. What does it mean? It means Ne the number of electrons is equal to the average or background number of electrons plus the perturbation. Perturbation is what physically you are causing and how the system response is going to be given out by the governing equations or the dispersion relation. Since number of electrons is a parameter which can directly be influenced by the moving of electrons, we take it to be perturbed. Then the velocity of electrons, when the electrons are trying to come back and restore charge neutrality, electron velocity is getting modified.

We write the electron velocity as U0 plus U1. U0 is the equilibrium velocity and U1 is the perturbed velocity. There will be a small deviation away from the background values. This small deviation is actually going to speak about the wave motion. The essence of this wave motion is going to be captured by this small deviations and the electric field is going to be influenced.

We will say E0 to be the electric field or equilibrium electric field and E1 to be the perturbed electric field which is actually being set up after you displace electrons. These parameters N0, U0 and E0 are generally referred to as equilibrium quantities. The importance of this equilibrium quantities is that they represent the basic state or they also represent the unperturbed state. They will remain a constant. That means that if you take a derivative of these quantities with respect to time, it will be 0.

These parameters N1, U1 and E1 are referred to as the perturbed quantities. Whatever the physical disturbance that we have created inside the plasma is inscribed in these three parameters. Perturbed quantities due to oscillations. Now for easiness, we will say that N1 will be much less than N0. These perturbations will be very small.

It is an obvious thing. We cannot take the perturbation to be equivalent to the background conditions and we have assumed a uniform or no change in the number densities and they are at rest before the oscillations are created. We say that delta N will be 0 and since we have assumed the velocity to be small, let us say you say that the equilibrium velocity is very small or 0. That means before the perturbation, things were stable or steady. And since at the time of perturbation or before the perturbation, the entire plasma is electrically neutral and there are no electric fields set up within the plasma. The electric fields are set up only after you displaced some electrons from here to there.

So, at equilibrium or unperturbed state, the electric field is also 0. What we can say is dou N by dou t is equal to dou U0 by dou t is equal to dou E0 by dou t would simply be 0. Let us say we say this condition as A and for reference, we have one equation 1, 2 and 3. Now, we have to substitute perturbation, let us say this is the perturbation. Let us say we call this as capital I and we substitute the perturbed variables in using I and A in equations 1, 2, 3.

So, this is a simple algebra. So, I am going to substitute the variables and write Me N e dou U1 by dou t is equal to minus e E1, we call this as 4 dou N1 by dou t plus N0 times del dot U is equal to 0, this is equation 5 and the Poisson equation epsilon naught times del dot E1 is equal to minus e times Ni minus N0 plus N1, this is equation number 6. What I have done is I have just substituted the perturbed variables into the governing equations. So, at equilibrium condition or before the disturbance has actually taken place, so the number of ions will always be equal to the number of electrons which is nothing but the unperturbed electron density which means number of ions will always be equal to number of electrons before the perturbation within the small region. So, which means that we can write N0 to be equal to Ni because epsilon naught times del dot E1 is equal to minus e N1. So, we call this equation as equation number 6.

So, you see these equations 4, 5, we do not need a number for this 4, 5 and 6. Now these quantities, the perturbed quantities E1, we have E1 here, we have N1 here and we have U1 here. So, all the perturbations are still existing in the respective equations. Now how are these variables changing with respect to time is actually equation. Now let us say we take these variables, the perturbed variables are oscillating sinusoidally.

So, we say that the perturbed variables are varying sinusoidally with respect to space. So, you can say U1 is equal to U1 exponential i kx minus omega t since it is velocity along the x direction, we say it as along i cap N1 is equal to N1 exponential i kx minus omega t, the electric field E is equal to E i kx minus omega t along i cap. So, let us say we call this set of equations as B. Now substituting set of equations in B into 4, 5 and 6. What are 4, 5 and 6? 4, 5 and 6 are the reduced governing equations.

Since these equations need to be solved, we assume a solution for the perturbed variables and our choice of solution for the perturbed variables which will actually solve a differential equation because a differential equation is something which expects a solution which is exponential in nature or sinusoidal in nature. So, that is why we have taken a solution which is like U1 is equal to U1 exponential i kx minus omega t, a sinusoidal solution will take the real values of this but nonetheless the solution will be valid for the equation number 4, 5 and 6. So, upon substitution we will get minus i Me omega U1 is equal to minus e E1 and minus i omega N1 plus N naught i k U1 is equal to 0 and epsilon naught i k E1 is equal to minus e N1. This is equation number 9. So, what we have done so far just recap is that we have understood that plasma facilitates several different types of wave motions.

We have understood what a wave is and how to characterize a wave. We have defined what is a phase velocity and what is group velocity and for example we have taken plasma oscillations which are inherent in every plasma as an example and we have imposed some conditions so as to develop a mathematical framework to derive a dispersion relation for plasma oscillations. So, based on these conditions we have taken the governing equations that we need to get the dispersion relation and in these equations there are several variables and we have picked up those variables which are most likely to be influenced by the perturbation that we create. So, we have split all these variables into a basic state variable plus perturbation and made some approximations which are valid for this type of oscillations and substituted the perturbed variables into the reduced governing equations. Once we do that we have assumed solutions for these differential equations which are oscillatory in nature, sinusoidal in nature and then we have substituted the perturbed variables which are sinusoidal back into the governing equations.

So, this is where we will stop today. In the next lecture we will try to see how these equations can be solved and the dispersion relation can be obtained. We will also see what is the consequence, what is the physical interpretation of the dispersion relation. Thank you.