Plasma Physics and Applications

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Week – 08

Lecture 40: Plasma as a fluid: Fluid Drift - II

Hello dear students. We are discussing plasma as a fluid and fluid drifts. In the last class, we have derived this expression which is called as the diamagnetic drift. We constructed a simple geometry and in that we have realized that the diamagnetic drift velocity will be charge dependent and will make the electrons and ions move exactly in opposite directions to each other. So we can start from the expression for the diamagnetic drift velocity Vd is minus del P cross B by qN B square. So we know that P is equals to N kT or we can use the relation P is equals to C N power gamma, gamma is the ratio.

So we can take del P which is C gamma N power gamma minus 1, derive this expression with P. So which follows del P by P is simply gamma times del N by N. So in the geometry we have del N. The concentration gradient is at a rate N prime along R cap, the magnetic field B is B along Z gap and the velocity V is obviously along phi gap.

Now this can be plus minus that means it can be along minus phi cap or along plus phi cap. So substituting all this we have del P is N kT gamma del N by N which implies del P is kT gamma del N. If we now substitute this into the expression for the diamagnetic drift velocity Vd is equals to minus gamma kT del N cross B by qN B square which is equals to minus gamma kT del N cross Z cap times B by qN B square. Simplifying further we can write Vd as plus minus which is just plus for positive charges, minus for negative charges and Z cap cross del N divided by qN B. So, I have just reverse the curl so that minus is taken care of.

So, this expression tells you that the diamagnetic drift velocity will be opposite for each. So, we can now write the diamagnetic drift velocity for the electrons as minus gamma kT electron temperature N cap by E the charge of electron N B along phi cap and the diamagnetic drift velocity for the ions will be plus gamma kT i divided by q N B times N cap along phi cap. So, this is the difference between electron diamagnetic drift velocity and the ion diamagnetic drift velocity. So, when these two velocities are different we can expect a current density which we will write as J dia as Ne Vdi plus Ne Vde or we can combine all that what we have written and simply write B cross del of Pi plus Pe divided by P square or using the notation that we have followed so far the diamagnetic drift velocities kT i plus kT times B cross del N divided by B square. So, this is the discussion about the diamagnetic drift.

$$v_D = \frac{-\vec{\nabla} P \times \vec{B}}{q n B^2}$$

$$P = nkT$$

$$P = C n^{\gamma}$$

$$\frac{\nabla P}{P} = \frac{C \gamma n^{\gamma - 1}}{C n^{\gamma}} \nabla n \Rightarrow \frac{\nabla P}{P} = \gamma \frac{\nabla n}{n}$$

$$\nabla n = n' \hat{r}$$

$$B = B \hat{z}$$

$$v = \hat{\varphi} \pm \Rightarrow -\hat{\varphi}$$
$$+\hat{\varphi}$$

$$\nabla p = nkT\gamma \frac{\nabla n}{n} \Rightarrow \nabla p = kT\gamma \nabla n$$

Just take a look back into what we have done so far. We have started with this expression for the diamagnetic drift velocity took the equation of state and wrote the pressure gradient in terms of concentration gradient. After making suitable substitutions we got this generalized expression for the diamagnetic drift velocity which is having plus minus symbol explicitly appear outside telling you that this will be positive for the ions and negative for the electrons. So, this positive or negative are going to be affected onto this the product of this curl because everything else is irrelevant only the direction of unit vector will accept this plus minus and tell you the direction of moment of electrons and ions which is actually opposite to each other. So, this diamagnetic drift velocity gives you a current density which is called as the diamagnetic current density.

$$v_D = -\frac{\gamma kT\, \vec{\nabla} n \times \vec{B}}{q n B^2}$$

$$= -\frac{\gamma kT\, (\nabla n \times \hat{z}) B}{q n B^2}$$

$$\boxed{v_D = \pm \frac{\gamma kT\, (\hat{z} \times \vec{\nabla} n)}{q n B}}$$

$$\left.\begin{aligned} v_{D_e} &= -\frac{\gamma k T_e\, n'}{e n B}\,\hat{\varphi} \\ v_{D_i} &= +\frac{\gamma k T_i}{q n B}\, n'\,\hat{\varphi} \end{aligned}\right\}$$

So, this is the case when we have velocity perpendicular to the magnetic field. Let us take another case which is called as the parallel drift or a case of drift velocity parallel to the magnetic field. So, magnetic field is as per the configuration is along the z direction and so far we have considered the perpendicular component of velocity and we have derived the diamagnetic drift velocity. So, we can say that if the pressure gradient is not perpendicular, so if this is the geometry just for remembering this is the direction of magnetic field that you have taken and this is the direction of del P this is B and this is del P. Now let us start with the same equation which is rho times now we are only talking about the parallel component dVz by dt plus V dot del Vz is equals to Q N ez minus dou P by dou z.

$$J_{dia} = nev_{di} + nev_{de}$$

$$= \frac{B \times \nabla (P_i + P_e)}{B^2}$$

$$\boxed{J_{dia} = (kT_i + kT_e) \frac{B \times \nabla n}{B^2}}$$

So, before this what should have been there is Q times N of course e plus V cross B minus del P. This is where you should have actually started and since you are only referring to the parallel component of velocity since the parallel component is in the same direction of the magnetic field this term becomes 0. This will sustain which is that is why you have taken the same direction and del P now you have made the density gradient along the direction of magnetic field. So, all of this tells you that the magnetic field is still on the z axis and the only difference is the velocity component along the magnetic field will not contribute the Lorentz term. So, here we again assume that the velocity is slow enough to ignore the second order term or that appears in the total derivative or the convective term.

<u>Drift velocity parallel to magnetic field</u>

$$\rho\left[\frac{dv_z}{dt} + (v \cdot \nabla) v_z\right] = qnE_z - \frac{\partial P}{\partial z}$$

v_z to be small such that convective term can be neglected

$$\frac{dv_z}{dt} \neq 0 \qquad \gamma = \frac{C_p}{C_v}$$

$$P = Cn^{\gamma}$$

$$\frac{\frac{dP}{dz} = C\gamma n^{\gamma-1}\frac{dn}{dz}}{P = Cn^{\gamma}}$$

But assume it to be not so small such that the velocity dVz by dt is 0. So, that means we can take this Vz of course to be small such that the convective term can be neglected. So, only the second order terms in velocity can be neglected. So, that means dVz by dt is not 0 or it need not be neglected. So, generally in plasma particles drift much faster along the magnetic field than at right angles to it.

$$\frac{1}{P}\frac{dP}{dz} = \frac{\cancel{C}\gamma n^{\gamma-1}}{\cancel{C n^{\gamma}}}\cdot\frac{dn}{dz}$$

$$\frac{1}{P}\frac{dP}{dz} = \frac{\gamma}{n}\cdot\frac{dn}{dz}$$

$$P = nkT$$

$$\frac{\partial P}{\partial z} = nk\cancelto{0}{\frac{\partial T}{\partial z}} + kT\frac{\partial n}{\partial z}$$

$$\frac{1}{nkT}\frac{\partial P}{\partial z} = \frac{\gamma}{n}\frac{\partial n}{\partial z}$$

$$\frac{\partial P}{\partial z} = \gamma k_B T\frac{\partial n}{\partial z}$$

$$\boxed{\nabla p = \gamma k_B T \nabla n}$$

$$\frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}$$

So, we can start from the equation of state and write it as P C N to the power of gamma N is actually representing the density. C is a constant. So, gamma know what is gamma? Gamma is the ratio of specific heats. If you are wondering what is gamma? Gamma is Cp by Cv. Now since you are taking plasma as a fluid all these variables or constants become relevant.

$$mn\left[\frac{\partial v_z}{\partial t}\right] = qnE_z - \frac{dP}{dz}$$

$$\boxed{\frac{\partial v_z}{\partial t} = \frac{q}{m}E_z - \frac{\gamma k_B T}{mn}\frac{\partial n}{\partial z}}$$

$$\frac{q}{m}E_z = \frac{\gamma k_B T}{mn}\frac{\partial n}{\partial z}$$

$$E = -\nabla\phi = -\frac{\partial \phi}{\partial z}.$$

$$\frac{q}{m}E_z = -\frac{q}{m}\frac{\partial \phi}{\partial z} = \frac{\gamma k_B T}{mn}\frac{\partial n}{\partial z}.$$

So, dP by dz is C gamma N to the power of gamma minus 1 dN by dz. So, dividing both sides of the equation by the equation itself. So, this is P is equals to C N to the power of gamma. If you do that you get 1 by P dP by dz is C gamma N to the power of gamma minus 1 by C N to the power of gamma dN by dz. So, which is 1 by P dP by dz is gamma by N times dN by dz.

Now substituting P is equals to N kT or N kBT. So, dou P by dou z is N k dou T by dou z plus kT dou N by dou z. We can take the temperature to be a constant. So, we can write 1 by N kT dou P by dou z is gamma by N dou N by dou z. This is what I have done as I have substituted P into this that is it.

$$-\frac{q}{\cancel{m}}\frac{\partial \phi}{\partial z} = \frac{\gamma k_B T}{\cancel{m} n}\frac{\partial n}{\partial z}.$$

$$\boxed{-q\frac{\partial \phi}{\partial z} = \frac{\gamma k_B T}{n}\frac{\partial n}{\partial z}}$$

$$q\phi + k_B T \ln n = k \qquad k_B T \ln n = k - q\phi$$

$$\boxed{n = n_0 \exp\left[\frac{-e\phi}{k_B T}\right]}$$

The remaining is the same. So, dou P by dou z is gamma kBT dou N by dou z. I think this step was not needed or we can write del P is gamma kBT del N or we can write del P by P is gamma times del N by N. So, we can substitute this into the momentum equation now. Which is mN dou V z by dou T equals to Q N z minus dP or dz or dou V z by dou T is equals to Q by mE z minus gamma kBT divided by mN dou N by dou z.

We have used what we had here that is it. So, this expression indicates the rate of change of velocity how it is affected by the electric field and the concentration gradient along the z axis. So, this is important along the z axis along the direction of magnetic field. So, now we have a combined situation due to the pressure gradient and the electrostatic force both of them. So, when the forces on the right hand side are balanced the fluid does not expect or experience any acceleration as such.

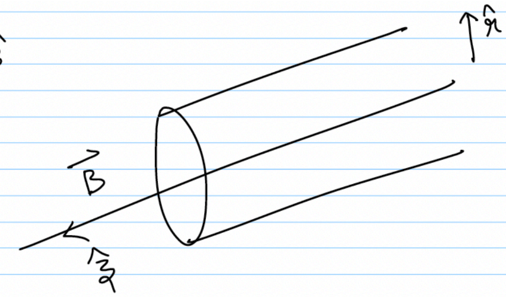
So, now let us assume a constant temperature along the magnetic field line and express the electric field as a gradient of the potential. So, Q by m so if there is if these two forces are balanced. So, Q by mE z is equals to gamma kBT by mN dou N by dou z. Now electric field is the negative potential gradient E is equals to minus del phi or if you are referring about one direction minus dou phi by dou z. So, we can write Q by mE z is equals to minus Q by m dou phi by dou z is equals to gamma kBT by mN dou N by dou z.

Example

Cylinder (r_0), $\vec{B} = B\hat{z}$

$n(r) = n_0 \exp\left(\frac{-r}{r^2}\right)$

$n_i = n_e = n_0 \exp\left(\frac{e\phi}{k_B T}\right)$



(i) $\vec{E} \times \vec{B}$

(ii) $\vec{v}_D$

$$\frac{\vec{E} \times \vec{B}}{B^2}$$

$$\frac{\nabla p \times B}{q n B^2}$$

So, minus Q by m dou phi by dou z is gamma kBT by mN dou N by dou z. We can get rid of this m here anyway let us say minus Q dou phi by dou z is gamma kBT by N dou N by dou z. Now let us integrate this equation and take gamma is equals to 1. What we will get Q phi plus kBT ln N is equals to k dou phi by dou z is phi dou N by N.

This will give you ln N. You take this dz outside then what you have is this is equals to constant. This minus will be taken care when you bring the other term to this side or if you write N, N if let us say N naught ln N is equals to this k is there. kBT ln N is k minus Q phi. And if you write N is equals to this k is N naught N naught exponential E phi by kBT. Or when you take electron Q can be written as minus E and this relation is the Boltzmann relation for electron when it is experiencing an external force due to phi.

So, this is some conclusion for the parallel drift when the particle or when the magnetic field is parallel to the pressure change or pressure gradient. So, what have we learned so far? We have seen two different situations in which the fluid experiences a drift. So, as long as you are considering plasma as a particle we have accounted for drift velocities, but when you have a fluid the additional force which comes into existence is the pressure force which is basically because of the concentration gradient. And we have now been able to implement even that force and obtain an expression for the drift velocity. We will take a small numerical just to appreciate these fluid drifts which is like this.

We will take a small example. So, we have a cylindrically symmetric isothermal plasma column of so we consider a cylinder of radius R naught in which plasma is filled. It has a magnetic field axial magnetic field which is B is equals to B along z cap. Now, you have a cylinder you have plasma inside it and you have this z cap direction which happens to be the direction of the magnetic field. Now, we have to think about what else? We have to think about this is r cap. So, radially outwards this is r cap rho cap depends on how you write your vectors.

And now we have to have radial density distribution we if it is there is no gradient in the density then there is no point. Now, the radial density distribution is given like N of R is N naught exponential minus R by R square and number of ions is equals to number of electrons is equals to N naught exponential E phi by k B T. So, what is it telling you? It is telling you that the number of electrons and ions are distributed in the Maxwell distribution function according to the Maxwell distribution function and along the radius of this cylinder the number of electrons or number of ions are having this exponential minus R by R square dependence. So, in this case what would be the you have to evaluate what is the E cross B drift velocity and also what is the diamagnetic drift velocity. So, these are the variables that are we given.

Let us try to understand the problem first. We have a cylinder in which there is an isothermal plasma. So, there is no variation with respect to temperature is constant and the dimensions of the cylinder are such that it is having a radius of R naught and along the axis of symmetry of this cylinder we have a magnetic field B. And how are the number of particles distributed? It is distributed according to this and with respect to the potential or the temperature they are following a Maxwell Boltzmann distribution function. Now we have to calculate these two drift velocities.

So, what do we need? We need we need the electric field E cross B by B square and Q this is del P cross B by Q n B square. Let us see now we know that n solution this is the solution we are trying to solve it. So, we know that n is n naught exponential E phi by k B T which is equal to n naught exponential minus R square by R naught square. So, there should be some potential that is responsible because we were not given the information about the electric field but we have a potential. So, this potential can be used to find out the electric field.

So, phi from this expression itself phi is equals to k B T by E ln n by n naught from this expression itself exponential taken to the other side will give you this ln. Now you know what is n by n naught from this the other half of this formula. So, we can write k B T by E minus R by R naught square. You see n by n naught is exponential minus R by R square. Ln of n by n naught simply removes that exponential and gives you minus R by R naught square.

Solution

$$n = n_0 \exp\left(\frac{e\phi}{k_BT}\right) = n_0 \exp\left(\frac{-r^2}{r_0^2}\right)$$

$$\phi = \frac{k_BT}{e} \ln \frac{n}{n_0} = \frac{k_BT}{e}\left(\frac{-r^2}{r_0^2}\right)$$

$$\vec{E} = -\nabla\phi = -\hat{r}\,\frac{\partial \phi}{\partial r} = \hat{r}\,\frac{k_BT}{e}\,\frac{2r}{r_0^2}$$

$$\vec{B} = B\hat{z}$$

$$\text{(i)}\quad \underbrace{\vec{E}\times\vec{B}}_{v_D} = \frac{\vec{E}\times\vec{B}}{B^2} = \frac{\hat{r}\,\frac{k_BT}{e}\,\frac{2r}{r_0^2}\times B\hat{z}}{B^2}$$

Now we know the electric field E is minus del phi which is minus R cap dou phi by dou R. We know the potential R naught is a constant. So, it is very easy to calculate this derivative. So, we have R cap if there is a minus inside this expression for phi. So, this minus will go R cap k B T divided by E 2 R by R naught square.

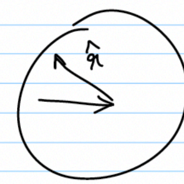
If you are wondering how I get how did I write R cap you have to go back and see what is the relation between the curl of direction of magnetic field and the direction of concentration gradient. So, this is the electric field. Now we have electric field and magnetic field of course magnetic field is B Z cap. So, the electric field is along R cap magnetic field is along Z cap and the concentration gradient is also along the R cap. So, we can now calculate the first one is E cross B drift velocity which is sorry E cross B this is V D E cross B by B square.

So, we can write all of this R cap k B T by E 2 R by R naught square cross B Z cap divided by B square. So, all the constants will come outside it is only the curl between R cap and Z cap which will give you minus phi cap. So, it will be V D will be 2 R by R naught square k B T by q B or E B along minus phi cap. This is the E cross B drift velocity. We are now calling the E cross B drift velocity with VE.

So, let us follow that VE. And the second question is the diamagnetic drift velocity V dia. The formula is minus del P cross B by q n B square. So, how do we get del P? We have n we can take del n which will be a representative of del P. So, using the notation that we have derived in the class. So, del P cross B can be we have to get del N is R cap dou N by dou R which is minus R cap 2 R by R naught square times N naught exponential minus R square by R naught square which is minus R cap 2 R by R naught square times N this is del N.

So, we know the expression for the diamagnetic drift velocity in terms of concentration gradient as k B T by q n B del N cross k. Substituting this we have V dia the diamagnetic drift velocity as phi cap k B T by q B have written electron temperature T e 2 R by R naught square. You see this is del N is minus R cap this is R cap the direction is R cap. The concentration the low pressure to high pressure is in this direction that is why I have written the gradient to be exactly opposite.

$$\boxed{v_E = -\frac{2r}{r_0^2}\frac{k_B T}{qB}\hat{\varphi}}$$



(ii) $$\underline{v_{Dia}} = -\frac{\nabla p \times B}{q n B^2}$$

$$= \nabla n = \hat{r}\frac{\partial n}{\partial r} = -\hat{r}\frac{2r}{r_0^2} n_0 \exp\left(-\frac{r^2}{r_0^2}\right)$$

$$\underline{-\hat{r}\times\hat{z}}$$
$$\hat{\varphi}$$

$$\nabla n = -\underset{\uparrow}{\hat{r}}\frac{2r}{r_0^2} n$$

$$v_{Dia} = \frac{k_B T}{q n B}\vec{\nabla} n \times \hat{z}$$

This is R cap and what is B is along Z cap. So, we have a curl of minus R cap cross Z cap which will give phi cap, but not minus phi cap. See you have phi cap and the remaining terms are just they will appear away. But what you see here is this is the diamagnetic drift velocity. You compare this with the E cross B drift velocity we have minus phi cap k B T e by q B 2 R by R naught square. When you compare these two things what you will realize is they are same in magnitude, but exactly opposite to each other in the direction.

$$\boxed{v_{Dia} = \hat{\phi}\,\frac{k_B T_e}{qB}\cdot\frac{2r}{r_0^2}}$$

$$\boxed{v_E = -\hat{\phi}\,\frac{k_B T_e}{qB}\cdot\frac{2r}{r_0^2}}$$

$$\boxed{v_{Dia} = -v_E}$$

$$\boxed{J_{Dia} = -\hat{\phi}\left(k_B T_i + k_B T_e\right)\frac{2r}{Br_0^2}\, n_0 \exp\left(\frac{-r^2}{r_0^2}\right)}$$

So, this is the conclusion that we have. So, for this type of symmetry or distribution of number of electrons and ions with respect to the potential as well as with respect to the radius with respect to the potential creates this electric field and with respect to the concentration creates this pressure gradient gives you an interesting result which is the diamagnetic drift velocity is exactly opposite to the E cross B drift velocity. You can go on and also evaluate the diamagnetic current density. You can try this as an exercise which is minus phi cap k B T i plus k B T e times 2 R by B R naught square times N naught exponential minus R square by R naught square. One extension of this all these discussions will also enable us to calculate the diamagnetic current density. So, this is about fluid drifts perpendicular to the magnetic field as well as parallel to the magnetic field. .