Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week – 08

Lecture 36: MHD Approximation -II

Hello dear students. In today's class we will continue our discussion on MHD approximation which is magneto hydrodynamics approximation which is essentially treating plasma as a magnetic fluid. So, it is MHD magneto hydrodynamics in which plasma is essentially treated as a fluid which is magnetized. So, we have defined these bulk parameters that means they will define the characteristic features of the entire plasma mass density, current density, mass velocity, total pressure and this is the total charge density. So, using this we have derived the charge conservation equation in which we realize that the rate of change of charge density with respect to time will be equated by the divergence of the current density. So, similarly we have what we have done is we have multiplied equation 1 and 2 with Qe and Qi and got this expression.

So, we can get the mass continuity equation which conserves the mass inside the plasma by similarly adding these 2 equations after multiplying with Me and Mi. So, we can start with the equations dou by dou T of Me Ne plus del dot Ne Me Ue is equals to 0. So, this is the continuity equation we have multiplied the equation with Me the mass of electron and similarly we can write an equation for the ion Mi Ni plus del dot. So, you must realize that in MHD approximation what we do is we write single equation of continuity or single equation for everything instead of 2 fluid equations.

$$P_{ei} = -P_{ie} \qquad m =$$

Mass density $\quad \rho_m = n_e m_e + n_i m_i \quad ; \quad \rho = n_e q_e + n_i q_i$ charge density.

Current density $\quad \vec{J} = n_e q_e \vec{u}_e + n_i q_i \vec{u}_i$

Mass Velocity $\quad \vec{U}_m = \dfrac{n_e m_e \vec{u}_e + n_i m_i u_i}{n_e m_e + n_i m_i} = \dfrac{\cancel{M}\cancel{L^{-3}} \times L T^{-1}}{\cancel{M}\cancel{L^{-3}}}$

Total pressure $\quad P = P_i + P_e$

So, what we can do is now let us say we call this equation as A and this one as B we can add these 2 equations A plus B. So, we can write dou by dou T the partial derivative with respect to time Ne Me plus Ni Mi plus del dot is equals to 0 or using the parameters that we have defined earlier we can write Ne Me plus Ni Mi plus the mass density del dot number of particles per unit volume multiplied by the mass which becomes mass per unit volume which is rho e Ue plus rho i Ui is equal to 0. Now we define the fluid mass velocity as U M which is equals to Ne Me Ue plus Ni Mi plus Ui divided by Ne Me plus Ni Mi which is still in the dimensions of velocity actually. So U M can be defined can be written as Ne Me the total density of the fluid rho M which is this part is nothing but rho e plus rho i. So, using this expression into this one what we can write is Ne Me Ue plus Ni Mi Ui is equals to U M rho M.

So, we can write the total continuity equation as dou by dou t of rho M plus del dot U M rho M is equals to 0. This one this is rho e. Let us slightly change the color scheme so that it becomes easy for you to understand if you are referring directly to the notes. This part becomes rho e plus rho i so does this part rho e plus rho i and we are defining rho e as rho M which is the entire density of the fluid. So, this becomes rho e plus rho i becomes rho M here and all of this becomes U M multiplied by the denominator.

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0 \quad \text{—— (1)}$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_i \vec{u}_i) = 0 \quad \text{—— (2)}$$

$$(1) q_e + (2) q_i$$

$$q_e \frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}_e q_e) + \frac{\partial n_i}{\partial t}(q_i) + \vec{\nabla} \cdot (n_i q_i \vec{u}_i) = 0$$

$$\underline{\vec{J}} = \underline{n e \vec{v}_d}$$

$$\frac{\partial}{\partial t}(n_e q_e + n_i q_i) + \vec{\nabla} \cdot (n_e q_e \vec{u}_e + n_i q_i \vec{u}_i) = 0$$

$$\boxed{\frac{\partial}{\partial t}(\rho) + \vec{\nabla} \cdot \vec{J} = 0} \quad \text{—— (I)}$$

<u>Charge conservation</u>

So, this is what we have here. So, this becomes dou by dou t of rho M plus del dot U M rho M is equals to 0. So, U M is the fluids mass velocity rho M is the density. So, we still have the same form the form has not actually changed much. But what we have done is instead of using two equations we now have only one continuity equation with this continuity equation is referred to as the mass continuity equation.

This equation speaks about the conservation of mass. So, how many equations we have got now? We have got two equations one for the charge conservation and another for the mass conservation. The other equation that is important is the equation of motion for the bulk plasma. This is equation of motion for bulk plasma. What does it mean? It means that we have to write F is equals to ma for the plasma which is not made up of two fluids it is only one fluid.

So, this is basically the momentum equation that we are very familiar with. So, we can get the momentum equation for a single fluid plasma if we add both the momentum equations of electron and ion. So, essentially we just have to add both the equations which is m e N e dou U e by dou t plus the advection term m e N e U e dot del times U e is equals to minus del p e plus q e N e times e plus U e cross p. The pressure force is taken into account and the Lorentz force is taken into account. So, we similarly write m i N i.

$$\frac{\partial}{\partial t}(m_e n_e) + \vec{\nabla}\cdot(n_e m_e \vec{u}_e) = 0 \quad \text{— (a)}$$

$$\rho_e + \rho_i = \rho_m$$

$$\frac{\partial}{\partial t}(m_i n_i) + \vec{\nabla}\cdot(n_i m_i \vec{u}_i) = 0 \quad \text{— (b)}$$

(a) + (b)

$$\frac{\partial}{\partial t}(n_e m_e + n_i m_i) + \vec{\nabla}\cdot(n_e m_e u_e + n_i m_i u_i) = 0 \longrightarrow \text{(a)}$$

$$\frac{\partial}{\partial t}(\underbrace{n_e m_e + n_i m_i}_{\rho_e+\rho_i}) + \vec{\nabla}\cdot(\rho_e u_e + \rho_i u_i) = 0 \longleftarrow$$

$$U_m = \frac{n_e m_e u_e + n_i m_i u_i}{n_e m_e + n_i m_i} \leftarrow \rho_e + \rho_i$$

$$\rightarrow U_m = \frac{n_e m_e u_e + n_i m_i u_i}{\rho_m}$$

The reason that I am writing these equations so many number of times is that by a practice you will remember these equations thoroughly. So, the idea is when I am writing these equations you should be able to understand what does each term represent and why the terms are added like the way they are done. So, the essential inference of all this mathematics is supposed to be understood, is supposed to be the context of this lecture. So, now we have these two equations which are basically the momentum equation for the electrons and ions or electron fluid and ion field which is we simply add these two equations. So, we will write this simple algebra which is dou by dou t of m e N e U e plus m i N i U i is equals to minus del of p e plus p i plus N e q e plus N i q i times e q e N e U e cross p plus q i N i U i cross p.

So, this is the electric field, the magnetic field. What we have done? We missed something. What is it? We missed or we neglected the convective derivative, the convective part of the total derivative, the convection term and we have only accommodated the first term which is the partial derivative of the velocity with respect to time. So, this is further simplifying it. We will write the left hand side as m e N e plus m i N i times dou by dou t of U m.

We know what is U m? Yeah, this is U m. U m is equals to minus del of p e plus p i plus this one, the total chart N e q e plus N i q i times e plus. So, this is J is equals to N e v, charge multiplied by the number density multiplied by the velocity. So, this is basically J cross B. This is for the electrons, plus J cross B for the ions or we can simply write it as N e m e plus N i m i dou by dou t of U m is equals to minus del of p e plus p i plus J cross B which is nothing but the idealized M H D equation for a single fluid plasma.

$$n_e m_e u_e + n_i m_i u_i = U_m \rho_m$$

$$\boxed{\frac{\partial}{\partial t}\rho_m + \nabla\cdot\left(U_m \rho_m\right) = 0}$$

Mass continuity equation

Conservation of Mass

Here we can, we have to, ideally we have to write this N e q e plus N i q i times e. But in an ideal case the plasma is electrically neutral and the electric field e is basically zero which means we can get rid of this term, we can make this term to be zero. So, idealized equation is, idealized equation for the bulk plasma, the momentum equation is basically this. What is the name of this equation? This is the equation of motion for the bulk plasma. Now, we have one more equation which is the rate of change of current density, the final equation of M H D.

So, the final equation describes the variation of current density with respect to time which will be less, we can start from the momentum equation itself which is m i N i times dou U i by dou t plus U i dot del times U i minus del p plus q i N i times e plus U j cross p. So, we can start with the momentum equation itself where it is U i. So, where i can be ion or electron. So, probably if it is confusing we can use a subscript of j because i generally denotes the ion. What we do is we take this equation and we multiply this equation by q e by m e and q i by m i and then add the two equations.

Equation of motion for Bulk plasma $\quad \vec{F} = m\vec{a}$

$$m_e n_e \frac{\partial \vec{u}_e}{\partial t} + m_e n_e (\vec{u}_e \cdot \vec{\nabla})\vec{u}_e = -\vec{\nabla} p_e + q_e n_e (\vec{E} + \vec{u}_e \times \vec{B})$$

$$m_i n_i \frac{\partial \vec{u}_i}{\partial t} + m_i n_i (\vec{u}_i \cdot \vec{\nabla})\vec{u}_i = -\vec{\nabla} p_i + q_i n_i (\vec{E} + \vec{u}_i \times \vec{B})$$

$$\frac{\partial}{\partial t}(m_e n_e \vec{u}_e + m_i n_i \vec{u}_i) = -\vec{\nabla}(p_e + p_i) + (n_e q_e + n_i q_i)\vec{E} + (q_e n_e \vec{u}_e \times \vec{B}) + (q_i n_i \vec{u}_i \times \vec{B})$$

$$(m_e n_e + m_i n_i)\frac{\partial}{\partial t}\vec{U}_m = -\vec{\nabla}(p_e + p_i) + (n_e q_e + n_i q_i)\vec{E} + (\vec{J}_e \times B) + (\vec{J}_i \times \vec{B})$$

$$\boxed{(n_e m_e + n_i m_i)\frac{\partial}{\partial t}\vec{U}_m = -\vec{\nabla}(p_e + p_i) + (\vec{J} \times \vec{B})}$$

So, I will write the two equations again q e N e times dou u e by dou t plus u e dot del times u e is minus del p plus q e N e times e plus u e cross p times q e by m e. So, this here on the left hand side you may be seeing that the mass is missing from here that is because it gets cancelled by the multiplying factor that you have which is q e by m e. So, similarly you can write q i N i times dou U i by dou t plus u e dot U i dot del U i is minus del p i plus q i N i times e plus U i cross p times q i by m r. We just add these two equations to get the rate of change of current density which will be written as dou by dou t of q e N e U e plus q i N i U i is equals to minus q e by m e del p e minus q i by m i del p i plus N e q e square by m e plus N i q i square by m i times the electric field plus N e q e square by m e times U e plus N i q i square by m i times U i cross p. So, what I have done I just go back to this I have just added these two took the factor q e by m e inside the bracket and this is what I get.

Now here in order to go further we need to impose the condition that the plasma is electrically neutral that means the number of charge carriers of ions and the number of charge carriers of let us say electrons are both equal or to say that the total positive charge is basically equal to the total negative charge. By writing this equation we are also accommodating the provision that if the positive ions are not singly ionized then the charge neutrality is maintained by having equal number of electrons to compensate for the effective positive charge. So, the total positive charge is equal to the total negative charge. Next we will find the current density J as N e U e q e plus N i q i U i and the mass velocity as N e U e m e plus N i U i m i divided by. So, using these two factors and this into this equation we can write it as dou J by dou t because you see the left hand side has the current density dou J by dou t is equals to minus q e by m e times del P e minus q i by m i times del P i plus N e q e square by m e plus N i q i square by m i times e plus u cross B plus q e by m e plus q i by m i times J cross B.

$$m_j n_j \left[\frac{\partial u_j}{\partial t} + (\vec{u}_j \cdot \vec{\nabla}) u_j\right] = -\nabla p + q_j n_j (\vec{E} + \vec{U}_j \times \vec{B})$$

$$\frac{q_e}{m_e} \; ; \; \frac{q_i}{m_i}$$

$$q_e n_e \left[\frac{\partial u_e}{\partial t} + (\vec{u}_e \cdot \vec{\nabla}) \vec{u}_e\right] = \left[-\vec{\nabla} p_e + q_e n_e (\vec{E} + \vec{U}_e \times \vec{B})\right] \times \frac{q_e}{m_e}$$

$$q_i n_i \left[\frac{\partial u_i}{\partial t} + (u_e \cdot \vec{\nabla}) \vec{u}_i\right] = \left[-\vec{\nabla} p_i + q_i n_i (\vec{E} + \vec{U}_i \times \vec{B})\right] \times \frac{q_i}{m_i}$$

So, this is the rate of change of current density with respect to time. So, this is the final MHD equation which tells you the rate of change of current density with respect to time. So, this is the final MHD equation which gives us the rate of change of current density with respect to tim

$$\frac{\partial}{\partial t}\left[q_e n_e \vec{u}_e + q_i n_i \vec{u}_i\right] = -\frac{q_e}{m_e} \vec{\nabla} p_e - \frac{q_i}{m_i} \vec{\nabla} p_i + \left[\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right] \vec{E}$$

$$+ \left[\frac{n_e q_e^2}{m_e} u_e + \frac{n_i q_i^2}{m_i} u_i\right] \times \vec{B}$$

$$q_i n_i = n_e q_e$$

$$\vec{J} = n_e u_e q_e + n_i q_i u_i$$

$$\vec{U}_m = \frac{n_e u_e m_e + n_i u_i m_i}{n_e m_e + n_i m_i}$$

e which is a single equation instead of two different equations. .

$$\frac{\partial J}{\partial t} = -\frac{q_e}{m_e}\vec{\nabla} p_e - \frac{q_i}{m_i}\vec{\nabla} p_i + \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$+ \left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)\left(\vec{J} \times \vec{B}\right)$$