Plasma Physics and Applications

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Lecture 35: MHD Approximation

Hello dear students. So, in continuation to our discussion on treating plasma as a fluid, we have understood how to write equations for plasma as a two fluid which means we have treated plasma as a combination of two intermixing and interacting fluids which are an electron fluid and an ion fluid. So, we have written equations for both electrons and ions separately. So, that is the basic understanding of plasma as a fluid that we have done so far. In today's class, we will try to understand what is called as MHD approximation. Now, so far what we know is plasma is a fluid of two entities, one an electron and ion.

So, what is the MHD and how MHD is going to be different from this two fluid approximation or two fluid theory? We call it as two fluid model of plasma. So, MHD stands for Magneto Hydrodynamics. So, you see hydrodynamics refers to the understanding of a fluid essentially and you are attributing magnetic nature or magneto you are calling it as a fluid which is conducting in nature because conducting fluid can only generate magnetic fields. So, it is basically the plasma is considered as a conducting fluid.

So, in this approximation, we will not treat plasma as two separate entities, we will treat plasma as a single fluid. So, how these things this approximation is going to modify the equations that we know is going to be the topic of discussion for today. So, under certain circumstances generally, it is more appropriate to consider the entire plasma as a fluid without differentiating between the electrons and ions. So, we will not differentiate electrons and ions as separate entities and we will not differentiate between different types of ions. We will consider all the different types of ions to be the same and we will not even differentiate between electrons and ions.

MHD Approximation

Magnetohydrodynamics

Fluid

— Fluid < ion, $\bar{e}$

$n_e, n_i, \vec{u}_e, \vec{u}_i$
$\vec{J}, \vec{E}, \vec{B}$
$m_e, m_i, \sigma, \ldots$
. . . . . .

Two fluid model of Plasma

conducting fluid

$\bar{e}$ & ions $\Rightarrow$ conducting

— Model highly conducting plasma

— Highly ionized

That means we are just looking at a fluid, we do not know if it is electrons or ions, we are just looking at the fluid as an entity which is conducting in nature. So, this approach of treating plasma as a conducting fluid is referred to as Magnetohydrodynamics MHD. So, MHD is a very important approximation or very important model to understand plasma. So, most of the highly conducting or high temperature plasmas can generally be easily understood using MHD equations or using MHD approximation. So, when is MHD valid? MHD is valid when we are trying to model when we are trying to model highly conducting plasma.

— MHD is a low frequency, long wave length approximation

— High conductivity, high degree of ionization

— Validity $\gg \lambda_D, r_e, r_i$

— Validity $\tau \gg \omega_p^{-1}, \Omega_i^{-1}, \Omega_e^{-1}$

— Assumes quasi neutrality

— $T_e = T_i$; Distribution $\Rightarrow$ Maxwellian

— MHD is very useful in Plasma instabilities

MHD can be very useful

— Solar wind
— Earth's magnetosphere
— Plasma turbulence
— — — -

So, if you are wondering what is MHD? MHD is just one way of describing plasma in which we treat the entire plasma as a conducting fluid, we do not differentiate between electrons and ions and we will write equations of motion only for the entire plasma not for electrons and ions separately. So, this method is very much appropriate when or under slowly varying conditions. So, things are not changing very fast and when the plasma is highly ionized, highly conducting plasma or you can say that highly ionized plasma which means the number density of the neutral species is very very small in comparison to the number density of the charge species. And when the electrons and ions are forced to act in unison due to a very frequent collisions or due to the influence of a very strong magnetic field, this MHD approximation becomes even more valid. So, that means electrons and ions are not allowed to act differently because of their charge or because of their mass.

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{u}_e) = 0$$

— C.E
— M.E
— P.E

$$m_e n_e\left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e\cdot\vec{\nabla})\vec{u}_e\right] = -\vec{\nabla}p_e + q_e n_e(\vec{E} + \vec{u}_e\times\vec{B}) + P_{ei}$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i\vec{u}_i) = 0$$

$$m_i n_i\left[\frac{\partial \vec{u}_i}{\partial t} + (u_i\cdot\nabla)u_i\right] = -\vec{\nabla}p_i + q_i n_i(\vec{E} + \vec{u}_i\times\vec{B}) + P_{ie}$$

$$\begin{array}{cc} e^-, & ion \\ \downarrow & \uparrow \\ i, & e^- \end{array}$$

Because of this frequent collisions it so happens that things are balanced out and electrons and ions are moving in a same way. So, that is when MHD is applicable. You cannot apply MHD to a very weakly ionized plasma, you will require high number densities wherein the individual nature or the characteristic behavior of electron is not exhibited or characteristic behavior of ion due to its charge or heavier mass is not exhibited rather both the things are acting in the same way. So, this is a very useful plasma approximation. So, what we will do is we will try to write down some conditions under which MHD is applicable or in terms of parameters that we know like plasma parameter, radius of gyration and frequency of gyration all these things.

So, let us say the approximations or MHD, the validity of MHD is subject to the satisfaction of the following conditions. So, what is MHD? The magnetohydrodynamic approximation is a low frequency is a low frequency or long wavelength approximation.

So, low frequency or long wavelength approximation it goes without saying or MHD requires high values of conductivity, high conductivity of plasma or very high degree of ionization which means the number density of charged particles is very high. And MHD is generally valid on length scales longer than device length. So, validity is where is it this MHD approximation valid? It is valid on length scales much larger than the device length.

$$P_{ei} = -P_{ie} \qquad m =$$

Mass density $\rho_m = n_e m_e + n_i m_i \quad ; \quad \rho = n_e q_e + n_i q_i$

Current density $\vec{J} = n_e q_e \vec{u}_e + n_i q_i \vec{u}_i$

Mass Velocity $\vec{U}_m = \dfrac{n_e m_e \vec{u}_e + n_i m_i u_i}{n_e m_e + n_i m_i} = \dfrac{\cancel{ML^{-3}} \times LT^{-1}}{\cancel{ML^{-3}}}$

Total pressure $P = P_i + P_e$

We know what is device length and the radius of gyration of electron or radius of gyration of ion. And most importantly it is valid on time scales longer than the inverse of plasma frequencies. When is it valid? It is valid when the time scale is much larger than the plasma frequency that means the time parameter associated with the plasma frequency. So, plasma frequency establishes how fast the charge neutrality will be established. So, this time scale over which this neutrality is established should be very small in comparison to the validity of the MHD approximation or you can also write it as omega I inverse or omega E inverse.

The MHD approximation assumes quasi neutrality which means in total the plasma is electrically neutral. And it also assumes that the collisions are frequent enough for the particle distribution to be Maxwellian and electron temperature becomes equal to the ion temperature. So, like I said electrons are not allowed to behave differently from the ions both of them are in the same Maxwellian distributions. So, the electron temperature becoming equal to the ion temperature is only by the means of very frequent collisions and the distribution of electrons or ions is essentially Maxwellian. We know what is the Maxwellian distribution function we have discussed this in the very beginning of the plasma physics course.

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{u}_e) = 0 \quad \text{——} \quad (1)$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i\vec{u}_i) = 0 \quad \text{——} \quad (2)$$

$$(1)\, q_e + (2)\, q_i$$

$$q_e\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{u}_e q_e) + \frac{\partial n_i}{\partial t}(q_i) + \vec{\nabla}\cdot(n_i q_i \vec{u}_i) = 0$$

$$\vec{J} = ne\vec{v}_d$$

$$\frac{\partial}{\partial t}(n_e q_e + n_i q_i) + \vec{\nabla}\cdot(n_e q_e \vec{u}_e + n_i q_i \vec{u}_i) = 0$$

$$\boxed{\frac{\partial}{\partial t}(\rho) + \vec{\nabla}\cdot\vec{J} = 0} \quad \text{——} \quad (\mathrm{I})$$

Charge conservation

And we will ignore some things let us we will not go into the limits of relativistic velocities or we will not involve any quantum mechanics in understanding or in formalizing the MHD equations. So, MHD by the nature of it mainly describes macroscopic force balance and which is imposing equilibrium and describes dynamically the dynamics of plasma reasonably well on very large scales. And most importantly MHD comes very handy if you want to understand the plasma instabilities. MHD is particularly useful is very useful in understanding in plasma instabilities. For example when is MHD a very good candidate to understand plasma is let us say you want to understand things like solar wind where MHD is particularly a success.

So, this is where MHD can be very useful. So, I am just naming some plasma systems where it is more appropriate to use MHD to in order to understand or in order to actually evaluate some physical parameters. Solar wind or earth's magnetosphere, plasma turbulence and so on. So, what have we learned so far just a quick introduction to MHD. MHD is one way of describing plasma or one way in which parameters related to plasma can be put together to understand how it is moving or how it is evolving with respect to time.

So, when you think of plasma what are the physical parameters which comes to your mind? It is basically the number of electrons or number of ions, the velocity of electrons or velocity of ions, the current density which may become relevant due to the differential motion of electrons and ions, the electric field which is applied or generated within the plasma, the magnetic field and the mass of the electrons and ions Me, Mi, the conductivity and so on. Now these are the dynamical variables. So, if you want to know the value of any of these you will need some equations and the way you write these equations is very important for evaluating or for finally calculating. So, what we have learned so far in our discussions is that we can treat plasma as a fluid, we can treat plasma as a two fluid entity where electron is a electron the entire electron density the population of electrons is a fluid and ions is also another fluid. Then we moved ahead and say if the plasma is highly conducting in nature and if the plasma is highly ionized then it is better to use better not to use a two fluid theory rather we combinedly say it is an entirely one fluid where electrons are not behaving differently in comparison to ions.

So, this idea is basically magnetohydrodynamics. This is establishing the basic definition of MHD and then we have seen all these conditions when the MHD approximation is valid or these conditions will tell you when you can treat plasma with an MHD model. And we also seen when is MHD particularly useful in which particular plasma system. Now what we will do is we will try to write or derive equations the what we know is the famous continuity equation, the momentum equation and the Poisson equation. All these equations we are we earlier derived all these equations for two fluids.

Now let us try to combine these equations and get a single equation. So, let us see we will start with the continuity equation first. So, for simplicity we will assume a single type of plasma with electron and a single ion species not multiple ion species it is a single ion species. So, generally we are used to writing the electron equation as dou NE by dou T plus del dot NE ue is equals to 0. What is this? This is the continuity equation which is telling that the rate at which the number density of electron changes with respect to time seems to be balancing the divergence of this NE ue.

So, this is the continuity equation for an electron. If you write the momentum equation for the electron we will write it as M e NE dou ue by dou T. Yeah, so what is this? This is the momentum equation which tells you the rate of change of velocity equal to the sum of forces. The forces that I have included here is the pressure force which is basically due to electrons as a fluid, the electrodynamic force which is due to the electric and magnetic fields and pei is basically the momentum transferred due to or by the electrons onto the ions when they collide with electrons with ions. So, similarly we can write the equations for ions.

So, dou NI by dou T plus del dot NI ui is equals to 0 and you have M i NI the momentum equation for the ions dou ui by dou T plus ui dot del. The advection term in the total derivative is minus del Pi plus qi NI times e plus ui cross e plus pi. So, pei and pie these two terms pei are the total momentum that is being transferred to electron and ion per unit volume per unit time by collisional interaction with the ion and electron. So, the momentum transferred to the electron and ion per unit time per unit volume by the ion and electron. So, it is simple momentum equation says the pressure whatever the pressure that is being built up by the electron plus the electrodynamic forces plus any additional momentum that is being transferred by the means of collisions by the ions onto the electrons.

And similarly when you write the equation for ions you will the momentum that is transferred to the ions by the electrons. Now what we will do is we will try to combine these equations and write a single equation. We will write a single equation for the conservation of mass as well as conservation of charge. Let us see how we do it. So, for a fully ionized plasma with equal number of electrons and ions we can simply write the momentum transferred between electron and ion will be equal to the momentum transfer between ion and electron.

Naturally we can say that this will be in the opposite direction to each other. So, when we take plasma to be completely ionized and electrically neutral we can say that the plasma is simply equivalent to a conducting to a highly conducting fluid. And the inertia of the plasma is because of the total mass of electrons and ions. So, when you write the mass of plasma it has to be we cannot write mass if we are writing a fluid it is better to write it in terms of mass density. So, in order to simplify the governing equations we will define our macroscopic parameters.

That means by considering the entire plasma as a single entity the entire fluid as a single entity we will define the macroscopic properties as the mass density rho m what is this? This is the mass so it has to be mass per unit volume. So, which is rho m is n e m e plus n i m i the volume information is in the n number of electrons per unit volume into mass and the current density. The current density of the entire plasma so which is J is equals to n e q e u e plus n i q i u i. And we will define what is called as the mass velocity which is u m as n e m e u e plus n i m i u i divided by m i. So, dimensionally it is like it is like m L power minus 3 into L T minus 1 divided by.

So, it is still the velocity and then we have the total pressure P is equals to P i plus P e. What are these parameters? These parameters are the macroscopic physical quantities which describe the entire plasma as a single fluid. We are used to using part of this for electron the remaining part for the ion. Now we have combined them and we have written a single macroscopic parameter for mass density, current density, mass velocity and total pressure. So, using these we will write combined equations of continuity of mass and charge that is the conservation of mass and charge separately.

Using these definitions we will derive combined equations for continuity of mass and charge. So, let us start with the continuity equation that is very familiar to us which is dou n e by dou T plus del dot n e u e is equals to 0. Let us say we write it as 1. Let us multiply this equation with q e before that let us write dou n i by dou T.

Plus del dot n i u i is equals to 0. Let us say we write this equation as equation number 6. Equation 1 is for the electron and equation 2 is for the ions. So, it is very simple the way it has been presented should be very easy to understand. There is no complicacy here. The name MHD appears to be very complicated, but what we are essentially doing is we are trying to combine equation of electron and ion together and write a single equation that is it nothing more than that.

The usefulness of these equations will be explained later. Let us multiply equation number 1 with q e and equation number 2 with q i and add them. Then we will write q e dou n e by dou T plus del dot n e u e q e plus dou n i by dou T and write q i here plus del dot n i q i u i is equals to 0. So, we can write it as dou by dou T of n e q e plus n i q i plus del dot n e q e u e plus n i q i u i is equals to 0. This is simple algebra there is no need for any sort of explanation here.

Using the predefined macroscopic variables defining the entirety of plasma we can write dou by dou T of rho plus del dot j is equals to 0. What is rho? Have we defined rho? What is rho? No, we have not defined. We have defined the mass density, but not the charge density. You see, maybe we can define the charge density also rho is n e q e plus n i q i. So, rate of change of charge density plus the divergence of the current density is equals to 0.

This is speaking something about charge conservation. So, this is the first equation of we can write let us say equation number 1 in Romans as the MHT. What is rho? Rho is the charge density charge per unit volume. Where is the information of volume here? In the n e, n e is number density per unit volume. Let us say number per unit volume and you are multiplying with charge.

So, charge per unit volume of electrons plus charge per unit volume of ions becomes the total charge per unit volume of the plasma. J is equals to n e V d. So, you must be familiar with this expression J is equals to n e V d, where the current density is number multiplied by the charge multiplied by the velocity. I have used the same thing here. So, this is the total charge conservation for the plasma.

So, we will take a break here and we will continue this discussion of deriving the rest of the MHD equation in the next lecture. Thank you.