Plasma Physics and Applications

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Lecture 34: Plasma as a fluid: Governing Equations

Hello dear students. So, in today's class we will try to understand the plasma behavior by considering it as a fluid, it is plasma as a fluid. So, far in our discussions we have taken plasma to be a particle and we have used the Lorentz force basically to obtain the trajectory of a particle when it experiences different types of electric and magnetic fields. But most of the times plasma can be treated as a fluid which is made up of two independent or interpenetrating fluid. One is called as an electron fluid and another is an ion fluid. So, this model of plasma in which you consider the entire plasma to be made up of two interpenetrating fluids is called as the two fluid theory.

Now, what we are going to do in today's class we will try to write down the complete set of equations which are required to understand plasma as a fluid. So, in our earlier discussions we have seen how individual equations describing the fluid motion can be derived. We are going to establish the complete framework of governing equations. If you remember our equation was very simple when you are considering plasma to be a fluid, to be a particle this is something like this.

So, it is m dv by dt is equals to the force due to the electric field plus the force due to the magnetic field. So, when you solve this equation depending on the type of field that we have taken we will get the position as a function of time. So, this is what our approach has been so far. Now, more importantly plasma most of the times can be easily explained by considering it as a fluid right. Now, how do we make the change? So, let us say we start from the basic equation.

So, we say F if it is made up of n number of particles F is summation of q summation over i q times e plus v i cross v right. Where we can use the mean velocity in this we can use what is called as the mean velocity as v is equals to sum over v i divided by delta n where delta n is the total number of particles. Now, as long as you are considering plasma as a particle you are addressing the individual behaviour of the particle. But when you consider it to be a fluid you no longer are looking at the individual behaviour rather you are looking at the collective behaviour of the entire fluid and trying to understand or try to write equations pertaining to the motion of the fluid. So, where delta n is the total number of particles in this Lorentz force equation which gives us the velocity v.

Plasma as a fluid

$$\left.\begin{array}{l} e^- \text{ fluid} \\ \text{ion fluid} \end{array}\right\} \text{Two-interpenetrating fluids} \Big\} \text{Two-fluid theory.}$$

Framework of governing equations

$$\bar{F} = e\bar{E} + e(\bar{v} \times \bar{B})$$

$$m\frac{dv}{dt} = \underbrace{e\bar{E} + e(\bar{v} \times \bar{B})}_{\text{Type of field}}$$

$$\downarrow$$

$$\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$$

So, if you consider the number of particles per unit weight volume then you can write delta n by delta v. This is the total number of particles and the number of particles per unit volume is this. Now you can write this the Lorentz force expression using this as q delta n times e plus v i cross v right. So, this is the electromagnetic force that the plasma will experience and if you want to use the expression the number of particles per unit volume you have to write n delta v times q times e plus v cross v. Or the electromagnetic force the particle or the plasma experiences per unit volume is equals to n q e plus v cross v.

So, this quantity that appears on the left hand side is called as the force density. We are just writing equations in a different notation. So, this is called as the force density or which is also equal to force per unit volume. Now so far we have been addressing the force just by this on the particle, but we are now calling it or quantifying it as force per unit volume. The reason is we are not considering the individual particle rather we are considering the volume of the fluid because we are referring to the entire volume of the fluid.

We want to see how much is the force that is experienced by a unit volume. So, similarly we can also write the momentum, momentum p is equals to summation over i m ui. You say mass is same for all the particles it can be written as m delta n v or m n delta v. So, this is now the momentum this is the total momentum. So, total momentum is equals to m n delta v v cap.

$$F = \sum_i q(\bar{E} + \bar{v}_i \times \bar{B})$$

Mean velocity $v = \dfrac{\sum v_i}{\Delta N} \longrightarrow$ Total no of particles.

$$n = \frac{\Delta N}{\Delta V}$$

$$\bar{F} = q\,\Delta N\,(\bar{E} + \bar{v}_i \times \bar{B}) \qquad \vec{F}$$

$$\bar{F} = n\,\Delta V\,q\,(\bar{E} + \bar{v} \times \bar{B})$$

$$\frac{\bar{F}_{EM}}{\Delta V} = nq\,(\bar{E} + \bar{v} \times \bar{B})$$

$\hookrightarrow$ Force density (or) $\dfrac{\text{Force}}{\text{Unit volume}}$

Momentum density similar to the force density we can write the momentum density is equals to momentum divided by the volume which is equals to m n v. So, now we have force density and momentum density. So, both of these can be used while using the plasma while using the fluid equations. So, in addition to the electromagnetic force when plasma is considered as a fluid it also experiences two additional forces which are called as the pressure force and viscous force. So, these two forces are in addition to electromagnetic force.

As long as we have considered plasma as a particle we do not have to deal with this type of additional forces because it is only the particle. But when you consider a fluid we know very well that differences of pressure at different points inside the fluid can exert a force which is called as the pressure gradient force or pressure force and the relative moment of fluid layers with respect to one another are influenced by what is called as the viscous force. So, if you just want to establish an expression for this it is like so if you have a neutral gas constituent of neutral gas molecules will exchange momentum when they collide. Now this momentum is what leads to the idea of pressure or the rate of momentum transfer to the walls of the container builds up the pressure or the force that is experienced by unit surface area on the walls is the pressure. Now the momentum that is lost when these particles collide with each other will be proportional to the relative velocity of the particles before the collision and after the collision.

$$P = \sum_i m\vec{u}_i = m \Delta N \vec{V} = mn \Delta V \vec{V}$$

Total momentum $= mn \Delta V \vec{V}$

Momentum density $= \dfrac{\text{Momentum}}{\text{Volume}} = mn\vec{V}$

— Pressure force
— Viscous force } In addition to E.M. force.

$f \propto (u - u_0)$

τ : Mean free time

$f \propto \dfrac{1}{\tau}$

$F = \dfrac{nm(u-u_0)}{\tau}$

So, we can write the F is proportional to u minus u naught and if you bring in tau to be the mean free time between the collisions then we can write tau is mean free time then we can write F is proportional to 1 by tau. So, in total we can write F is equals to Nm u minus u naught divided by tau. So, this rate of change of momentum is basically the force. Now this can be added to the force equation that we had written earlier. So, we can write m n dou u by dou t plus u dot del dot u is equals to q n e plus v cross v minus m n mu u minus u naught.

Mu is coming from the inverse of the tau minus del p. So, this is the equation which is valid for the fluids. So, this is the term which takes care of the electromagnetic force. This is the term which takes care of the viscous force and this is the pressure force. And whatever that appears in the square bracket as we have already discussed this is the total derivative.

So, if you look at this equation on the left hand side you have the rate of change of momentum on the left hand side you have the acceleration term but written by using a total derivative because we want to account for the motion of the fluid also while following the motion. So, the acceleration of the fluid element or fluid volume is equals to the effects of imposed by the electromagnetic force, viscous force and the pressure force. So, if you take this we can now go ahead and see how will be the full set of equations that will appear when we consider plasma as a fluid. So, now plasma is going to be called as a mixture of two interpenetrating fluids. For simplicity we will consider the plasma to be fully ionized and that means that there are only electrons and ions and no neutral species are present.

Then we consider plasma to be isotropic so that means that the plasma density is same in all directions. So, we can say for simplicity that there will not be pressure gradients in the plasma and we consider plasma to be collisionless. So, what does it mean? It means that we do not have to worry about the collision term. Collision term can be neglected. So, when we want to establish the complete set of governing equations for plasma as a fluid we need to make these assumptions.

$$\underbrace{mn\left[\frac{\partial u}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u}\right]}_{\text{Total derivative}} = \underbrace{qn\,(\bar{E}+\bar{v}\times\bar{B})}_{\text{Electromagnetic force}} - \underbrace{mn\nu(u-u_0)}_{\text{Viscous force}} - \underbrace{\nabla p}_{\text{Pressure force.}}$$

Plasma

- Two interpenetrating fluids
- e^-, ions & no neutral species
- Isotropic $\Rightarrow \nabla p = 0$
- Collisionless $\Rightarrow$ Collision term can be neglected

What are the intuitively we can make a guess how many equations or what are the equations that we want to involve. Let us say we want to conserve mass when the fluid moves from one place to another place. We have to conserve momentum and then we need the ideal gas equation and then since plasma is electrodynamic in nature plasma as a fluid also can be influenced by the electromagnetic forces electromagnetic fields. So, it is invariant that we have to bring in the Maxwell equations. So, we have already discussed what is the need for considering plasma as a fluid.

So, just to have a repetition. So, plasma when it is considered as a particle we always thought this type of fields let us try this type of different electric and magnetic fields and then try to understand the plasma motion. But at the end we realize that generally the fields that we try to impose are not prescribed. So, we do not prescribe what type of fields will exist but they are actually determined. So, plasma itself determines what kind of fields will be there and it is a self consistent motion.

charge density $\rho = n_i q_i + n_e q_e$ —— (1) S

current density $J = n_i q_i v_i + n_e q_e v_e$ —— (2) V

- Conserve mass
- Conserve Momentum
- Ideal gas equation
- Maxwell equations

(V)

Momentum equations

(6)

$$m_e N_e \left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e \cdot \vec{\nabla}) \vec{u}_e \right] = - \vec{\nabla} p_e + q_e N_e \left(\vec{E} + \vec{u}_e \times \vec{B} \right) \quad \text{— (3)} \quad (V)$$

$$m_i N_i \left[\frac{\partial \vec{u}_i}{\partial t} + (\vec{u}_i \cdot \vec{\nabla}) \vec{u}_i \right] = - \vec{\nabla} p_i + q_i N_i \left(\vec{E} + \vec{u}_i \times \vec{B} \right) \quad \text{— (4)} \quad (V)$$

$B = (B_x, B_y, B_z)$

$u_e = (x, y, z)$

$u_i = (x, y, z)$

$E = (E_x, E_y, E_z)$

Maxwell Equations

$$(V) \quad (\vec{\nabla} \times \vec{E}) = - \frac{\partial \vec{B}}{\partial t} \quad \text{— (5)}$$

$$\epsilon_0 (\vec{\nabla} \cdot \vec{E}) = N_i q_i + N_e q_e \quad \text{— (8)} \quad (S)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{— (6)} \quad (S)$$

$$\frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) = (N_i q_i u_i + N_e q_e u_e) + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{— (7)} \quad (V)$$

$\nabla \cdot B = 0$

So, particle theory may not actually be suitable for the entire description of plasma. So, this is the set of equations that we need. So, let us say we write these conservation laws and equations mathematically. So, let us say we assume the charge density of plasma as rho is equals to Ni Qi plus Ne Qe. What is this? This is the charge density.

Since plasma is neutral the total charge density that is available in the plasma is rho is equals to Ni Qi plus Ne Qe. Let us say we call this equation as 1. Since it is moving there are currents inside the current density let us say we call this J which is equals to Ni Qi Vi plus Ne Qe Ve. N is the number of particles per unit volume, Qi is the cumulative charge of ions, Vi is the velocity of ions plus Ne Qe Ve same terms for the electrons. These are two equations then we will need the momentum equations.

dot del times Ue is equals to minus del pe plus Qe Ne e plus Ue cross pe. What is this? This is the momentum equation for an electron which is the left hand side you have the rate of change of velocity and on the right hand side you have all the forces which are responsible for this rate of change of velocity. Then we have to write a similar equation for an ion. So, we will write Mi Ni dou Ui by dou t plus Ui dot del times Ui that direction term is equals to minus del Pi the pressure force plus Qi Ni e plus Ui cross pe. Let us say we call this equation as equation number 3, this as equation number 4.

Now these are the momentum equations for electron and ion. Then we have to write the Maxwell equations. Maxwell equations tell you how the charge densities and the fields and the currents are connected with each other. So, Maxwell equations are del cross E is equals to minus dou B by dou t del cross del dot B is equals to 0 and 1 by mu naught del cross B is equals to Ni Qi Ui plus Ne Qe Ue plus epsilon naught dou E by dou t. What else you have? You have one more equation which is del dot E the divergence of the electric field is equals to del dot E by epsilon naught times del dot E is equals to Ni Qi plus Ne Qe.

Let us say we call this equation as equation number 5, 6, 7 and 8. We have four Maxwell equations. I have just written them here. Then what else you have? You have to write the equation pertaining to the conservation of mass which is the continuity equation. So, we have to write the continuity equation for the electron fluid as well as for the ion fluid.

The continuity equation which is dou Ne by dou t plus del dot Ne Ue is equals to 0. We call this equation as equation number 9, dou Ni by dou t plus del dot Ni Ui is equals to 0. We call this as equation number 10. We already have 10 equations. In addition, we will need the thermodynamic properties to be considered or the kinetic properties of the fluid which is just a fluid kinetic properties which can be addressed by the simple equation P is equals to nkT.

So, electron fluids pressure is equals to kB electron temperature and Pi is equals to kB Ti Ni. Let us say this is equation number 11, equation number 12. They can also be written as Pi is equals to C rho i to the power of gamma or Pe is equals to C times rho e to the power of gamma. C is a constant and gamma is the ratio of specific heats. So, let us say we call them as equation number 13 and 14.

So, we now have 14 equations. So, now let us see if we want to solve them. What is the basic approach? So, basic approach is you can solve them, you can obtain few variables, you can substitute those variables into other equations and then obtain those unknown variables. So, ideally we will expect that we will need a condition to be satisfied always when you are solving some equation. So, we will always read. So, number of variables must always be equal to the number of equations or we are more familiar to call this as unknowns, number of unknowns should be always be equal to the number of equations.

Continuity equation

$$\frac{\partial N_e}{\partial t} + \nabla \cdot (N_e u_e) = 0 \quad \text{— (9)}$$

$$\frac{\partial N_i}{\partial t} + \nabla \cdot (N_i u_i) = 0 \quad \text{— (10)}$$

(V)

$N_i + N_e = N$

$N_i = (N - N_e)$ into (10)

Kinetic properties (1)

$$P_e = k_B T_e N_e \quad \text{— (11)}$$

$$P_i = k_B T_i N_i \quad \text{— (12)}$$

$$P_i = C \rho_i^{\gamma} \quad \text{— (13)}$$

$$P_e = C \rho_e^{\gamma} \quad \text{— (14)}$$

(5)

$$\frac{\partial (N - N_e)}{\partial t} + \nabla \cdot (N - N_e) u_i = 0$$

(9)

Unknowns ← Number of Variables = Number of equations

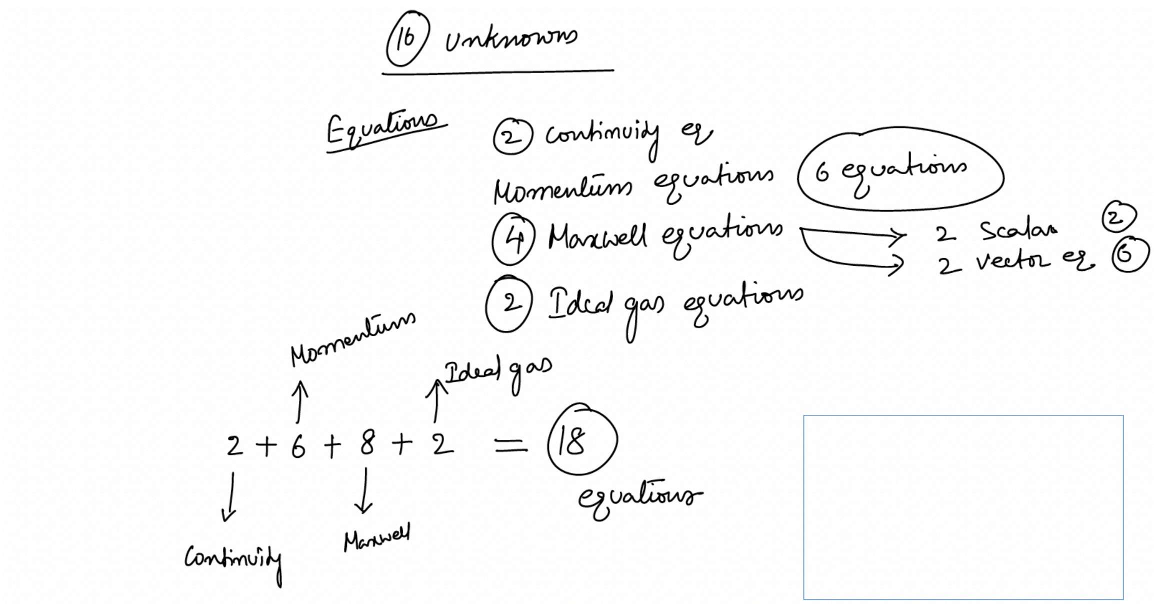
Vector Unknowns $\Rightarrow \vec{v}_e, \vec{v}_i, \vec{E}, \vec{B}$

Scalar Unknowns $\Rightarrow n_e, n_i, P_e, P_i$

Now, let us try to count how many equations we have and how many variables do we have? So, we have how many vector variables. So, you have the velocity, ue will have three component velocity, ui will also have three component velocity, let us say x component, y component, z component, x component, y component, z component. The electric field will have Ex, Ey, Ez, the magnetic field will also have Bx, By, Z. So, the number of variables, these are the vector unknowns, we are calling them as the vector unknowns. We will say vector unknowns are the electron fluid velocity, the ion fluid velocity, the electric field and the magnetic field.

Similarly, we have scalar unknowns. What are they? We have Ne, Ni, Pe, Pi. So, equation numbers 3, 4, 5, 6 are vector equations and 7, 8, 9, 10, 11, 12, 13, 14. Equation 1 is a scalar equation. Let us say we write it here, it will be easy.

This is a scalar equation. Since the velocity is there, this is a vector equation. This is a vector equation. These are basically vector equations and here velocity is there. These are also vector equations.



These are scalar equations. But in a sense, we have these variables which are scalar and vectors. So, in total we have 16 unknowns. So, let us see how many we have. So, we have 3 component magnetic field, 3 component velocity, ion velocity, 3 component electron velocity and 3 component electric field. So, we have 12 variables from here and then we have the pressure of electron, pressure of ion, number of electrons and number of ions.

So, hence we have 16 unknowns in total. But how many equations do we have? 16 unknowns we have and the number of equations. We have 2 continuity equations. We have momentum equations. So, this is a vector equation which upon expanding can be written into 3 component form. So, we have 6 equations which are from the momentum equation.

Then we have 4 Maxwell equations. Then we have 2 ideal gas equations. And within Maxwell equations, we have 2 scalar equations and 2 vector equations. Because you go here, so all of these are not vector equations. So, in this there are 2 equations which have divergence. They are the scalar equations and this is the scalar equation and this is the vector equation and this is the vector equation.

So, 2 vector equations upon splitting will give you 6 equations. This is equal to 6, this is plus 2. So, in total 2 for the continuity, 6 for the momentum, 8 for the Maxwell equations plus 2 for the ideal gas equation. So, we have 18 equations in total. So, this is just for the sake of remembering continuity momentum Maxwell ideal gas.

$$\nabla \cdot B = 0$$

$$\underbrace{\nabla \cdot (\nabla \times E)}_{0} = \nabla \cdot \left(-\frac{\partial B}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \cdot B) = 0$$

(5) & (6)

(1) ⇒ Considered

17 equations

16 unknown

16

(1) Continuity equation can be dropped

But in contrast, we see that there are only 16 unknowns. So, we have 2 extra equations. Let us see if there is any redundancy in the number of in the equations that we have considered. The point is, is there any equation which is conveying the same message or same information as the other equation? So, we have to see if there are any 2 equations which are conveying the same message. In that case, we can drop one equation and we can bring this equality between the number of equations and the number of variables or the number of unknowns the same.

Now, if you see this equation del dot B is equal to 0. What it conveys is that the divergence of magnetic field is always 0 or it also says that there is no way that you can have magnetic monopoles. You take these 2 equations, these 2 Maxwell equations, we can establish a very interesting relation. So, del dot B is equal to 0. We know that. Now, let us say del dot if I take a divergence of del dot del cross E which will be equal to del dot minus dou B by dou t or minus dou by dou t of del dot B which is equal to 0.

So, having this equation del cross E is equal to minus dou B by dou t which is equation number 1, equation number 5, the first Maxwell equation that we have written. So, the idea is if you take a del dot if you take a divergence of this, you are going to using vector identities, we are going to establish that del dot B is equal to 0 because the divergence of curl is 0. You come back here. So, this even if you do not know this, you know that this is 0.

So, del dot B has to be 0. So, equation number 5 is conveying the same message as equation number 6. So, in that case, we can keep only 1 Maxwell equation among these equation number 5 and 6. So, only 1 equation can be considered. That means now we are in a situation we have 16 unknowns and 17 equations.

We have to get rid of 1 more equation. Now, if you go here, if you take this continuity equation, which is this. So, you take the total number Ni plus Ne as N and substitute Ni is equals to N minus Ne into equation number 10. So, what we will get is dou Ni will get is dou of N minus Ne by dou T plus del dot N minus Ne times ui is equals to 0. So, if you take the total number to be a constant, you will get the same equation, you will get equation number 9 from this. Which means that out of the two continuity equations, we can only take one equation.

So, that means one continuity equation can be dropped. So, in total we now have 16 equations which are matching with the 16 unknowns. Let us see what those 16 equations are so that we can summarize this discussion. The 16 equations are we will need this charge density and the current density. We will need the momentum equation in for ions and electrons and out of this we will probably take this equation, this equation and this equation. How many are there? 1, 2, 3, 4, 5, 6, 7 and these two equations are actually equivalent to 6 equations.

6, 8, 9, 10, 11 from here and out of these we will take only one equation and out of these two we will take one equation. That is it. So, in total we have the equal number of equations and equal number of variables. So, this describes the complete fluid system using these equations.

So, this is what it is. We now have the complete set of governing equations which can describe the moment of fluid which is the plasma. Moment of a fluid which is electrodynamic in nature, so it demonstrates the properties of a fluid and at the same time it also has an ability to be influenced by the electric and magnetic fields.