Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week – 07

Lecture 33: Plasma as a Fluid: Fluid Equation -II

Hello dear students, we will continue our discussion on continuity equation and how we can derive different forms of the continuity equation by using different control volumes. So far in our discussion we have seen the rate of change of mass into the Eulerian control volume can be written as dM by dT is equals to minus dou by dou x of rho u delta x delta y delta z minus dou by dou y of rho v delta x delta y delta z minus dou by dou z of rho w delta x delta y delta z. We can divide this expression dividing by delta x delta y delta z which is delta v which will become 1 by delta v dM by dT is equals to minus dou by dou x of rho u minus dou by dou y of rho v minus dou by dou z of rho w. The mass that is the independent variables here are x, y, z and T. We are evaluating the rate of change of mass with respect to only time. We can write it as dou M by dou T into 1 by delta v.

Since mass divided by volume is density we can write it as dou rho by dou T is minus dou by dou x of rho u plus dou by dou y of rho v plus dou by dou z of rho w or the partial change of density with respect to only time will be equal to all of this can be written as the divergence of this scalar del dot rho u or if you have to write u as the vector it can be u. This can be minus del dot rho u where u is the three dimensional component of velocity ui plus vj plus wk bar. Now, we have to just recall del is i cap dou by dou x plus j cap dou by dou y plus k cap dou by dou z. You multiply with a rho this expression take a dot product of this you will get this relation.

So, taking this relation out we can write dou rho by dou T is minus del dot rho u. The rate of change of density with respect to time is equal to the divergence of the mass flux. So, this is one representation of the continuity equation. We can slightly get a different form of this by using a vector identity. Let us say we can write del dot rho u is equals to rho times del dot u plus u dot del rho because rho is a scalar.

$$\frac{dM}{dt} = -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z - \frac{\partial}{\partial y}(\rho v)\delta x \delta y \delta z - \frac{\partial}{\partial z}(\rho w)\delta x \delta y \delta z.$$

Dividing by $\delta x \delta y \delta z = \delta V$

$$\frac{1}{\delta V}\frac{dM}{dt} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) \qquad x, y, z, t$$

$$\frac{1}{\delta V}\frac{\partial M}{\partial t} = \frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \qquad \frac{M}{V} = \rho$$

$\square \ \frac{\partial \rho}{\partial t}$

$$\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \rho \vec{U}} \qquad \vec{\nabla} \cdot (\rho \vec{u})$$

$$\rho \vec{U} = \rho(u\hat{i} + v\hat{j} + w\hat{k})$$
$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$
$\odot$

So, we can write minus del dot rho u as minus rho times del dot u minus u dot rho. So, this is what we have on the right hand side of this equation. What is this? This is the equation of continuity. So, we can use all of this on the right hand side of this equation. So, this expression is only valid when you have del rho that is like rho being a scalar del rho the gradient of this density is non-zero.

That means, you expect the density should be changing within that. Then that means, that you are referring to different points in the flow. So, let us substitute all of this is minus. So, we can rewrite it as dou rho by dou t plus u dot del rho is equals to minus rho times del dot u. What have I done anyway? You look at this expression, I have taken the right hand side of the equation of continuity and using a simple vector identity I have expanded this.

$$\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{U})}$$

Equation of continuity.

$$\vec{\nabla}\cdot\rho\vec{U} = \rho(\vec{\nabla}\cdot\vec{U}) + \vec{U}\cdot\vec{\nabla}\rho$$

$$-\vec{\nabla}\cdot\rho\vec{U} = -\rho(\vec{\nabla}\cdot\vec{U}) - \vec{U}\cdot\vec{\nabla}\rho$$

(x, y, z, t)

$\frac{\partial}{\partial t}$

$$\frac{\partial\rho}{\partial t} = -\rho(\vec{\nabla}\cdot\vec{U}) - \vec{U}\cdot\vec{\nabla}\rho$$

$\frac{D}{Dt}$

$$\underbrace{\frac{\partial\rho}{\partial t} + \vec{U}\cdot\vec{\nabla}\rho} = -\rho(\vec{\nabla}\cdot\vec{U})$$

$$\boxed{\frac{D\rho}{Dt} = -\rho(\vec{\nabla}\cdot\vec{U})}$$

Velocity divergence form of continuity equation.

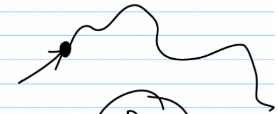
That is del is operating del as a vector operator operating on this product and so this is del first operates on the velocity because it is a vector it is a scalar product and when it operates on this scalar this is all you get. Now I have taken this term to the other side multiplied with a minus the entire expression becomes this and it turns out this expression has a term which is something related to the total derivative. So, I have substituted all of this in the place of this on the right hand side and pulled this term which is this term to the other side or I have taken this term u dot del rho to the left hand side. Now what I have on the left hand now is the partial change in the density with respect to time plus an advection term. So, this is from the standard definition of the total derivative.

So, total derivative is equals to the partial derivative of the quantity with respect to time plus an advection term which accounts for change along the motion. This can be written as d rho by capital D rho by dt is minus rho times del dot u. This expression is called as the velocity divergence form of the continuity equation. This is also the continuity equation but this has a name because you have the velocity divergence appearing on the right hand side. This is earlier form had the divergence of mass flux but this has the divergence of velocity.

Now you look at this term why I have written the total derivative. Total derivative as long as your independent coordinates are x, y, z and t. These are the independent coordinates. As long as you are trying to measure change with any one of these you have to think of a partial derivative. But if you are trying to measure change as a function of time and also as this control volume is moved from one set of space coordinates to another set of space coordinates there is a change.

Lagrangian Control Volume

$$\delta M = \rho \delta V = \rho \delta x \delta y \delta z$$



$$\frac{D}{Dt}$$

$$\frac{1}{\delta M}\frac{D}{Dt}\delta M = \frac{1}{\rho \delta V}\frac{D}{Dt}\rho \delta V \qquad \text{— (1)}$$

$$= \frac{1}{\cancel{\rho} \delta V}\cancel{\rho}\frac{D \delta V}{Dt} + \frac{1}{\rho \cancel{\delta V}}\cancel{\delta V}\frac{D\rho}{Dt} = 0$$

$$\underbrace{\frac{1}{\rho}\frac{D\rho}{Dt}} + \underbrace{\frac{1}{\delta V}\frac{D\delta V}{Dt}} = 0 \qquad = \vec{\nabla}\cdot\vec{U} \qquad \text{— (2)}$$

$$\frac{1}{\delta x \delta y \delta z}\frac{D}{Dt}\delta x \delta y \delta z = \frac{1}{\delta x}\underbrace{\frac{D}{Dt}}\delta x + \frac{1}{\delta y}\frac{D}{Dy}\delta y + \frac{1}{\delta z}\frac{D}{Dz}\delta z$$

This change that is brought in because of the change in space coordinates is represented by this and because of the time that elapses from one point to another point is represented by this. A combination of these two is called as the total derivative. Earlier we had this equation of continuity which is written in terms of partial derivative. This form is using the total derivative. This is also called as the velocity divergence form of continuity equation.

We are still left with the task of deriving the continuity equation using Lagrangian control volume. What is the difference between the Lagrangian control volume and the Eulerian control volume? The Lagrangian control volume is something which is moving with the fluid. It is not at rest. Now it is moving with the fluid. Earlier we had the control volume fixed and the budgets were calculated with respect to that volume.

How much is going in and how much is getting out. Now for the sake of Lagrangian control volume what we will do is we will consider a small mass of particles delta M whose mass is delta M and we can write it as rho delta X delta Y delta Z. What about this mass of particles? This is the volume itself. You have this small packet of the fluid. Now you are looking at this fluid this point and this mass.

This mass is going everywhere like it is following it is moving along the fluid. Now since now the moment is involved we cannot think of finding out the change only with respect to time we have to think of invoking the total derivative. The relevant derivative now is d by dt that is the basic difference. You see you go back here we had the control volume which is fixed. Now you assume that X, Y, Z are constant and you are only changing something let us say density only with respect to time that is why you had the partial derivative in place.

But now since it is moving everywhere you have to think of the total derivative in instead. So, for the conservation of mass ultimately when you talk about deriving the continuity you are only referring to the conservation of mass. So, for the conservation of mass let us say that this delta M the mass needs to be conserved. So, whatever the changes delta M will remain a constant. So, that means that we can write a fraction let us say delta M T by dt of delta M which is 1 by rho delta V d by dt of rho delta V.

$$\cdot A \qquad u_A = \frac{Dx}{Dt}$$

$$\cdot B \qquad u_B = \frac{D}{Dt}(x+\delta x)$$

$$\delta u = u_B - u_A = \frac{D}{Dt}(x+\delta x) - \frac{D}{Dt}x = \frac{D}{Dt}\delta x$$

$$\frac{1}{\delta V}\frac{D}{Dt}\delta V = \frac{1}{\delta x}\frac{D}{Dt}\delta x + \frac{1}{\delta y}\frac{D}{Dt}\delta y + \frac{1}{\delta z}\frac{D}{Dt}\delta z$$

$$= \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}$$

$$\underset{\delta \to 0}{\text{Lt}} \quad = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{1}{\delta V}\frac{D}{Dt}\delta V = \vec{\nabla}\cdot\vec{U} \qquad \text{—— } (3)$$

So, we can write it as 1 by rho delta V times I will keep rho outside and T of delta V by delta T plus 1 by rho delta V and keep delta V outside d by dt of rho is equals to expect that this to be 0 for the conservation to work. So, this can be cancelled and this equation can be simplified as can be simply written as 1 by rho d rho by dt plus 1 by delta V d delta V by dt is equals to 0 fair enough. So, this is delta V is the control volume. So, delta V is delta X delta Y delta Z. So, we will expand it again.

So, 1 by delta X delta Y delta Z times d by dt of delta X delta Y delta Z is equals to 1 by delta X d by dt of delta X. What happened to delta Y and delta Z? The moment you keep them outside and you operate the derivative on to delta X, this will these two terms will get cancelled with the denominator that appears outside the derivative. So, that is a simple step that I am not writing here, but you can follow it plus 1 by delta Y d by dy of delta Y plus 1 by delta Z d by dz of delta Z. Now, this is one expression. So, we have this expression plus all of this and the summation should be 0.

Now, let us say the control volume that we have considered is moving with respect to the fluid. It is moving. So, let us say the control volume at a point A has a velocity UA and at another point B has a velocity UB. So, UA can be written as dx by dt and UB subsequent point in time is d by dt of X plus delta X at a subsequent point. So, delta U the change in the velocity delta U is nothing but UB minus UA which is basically d by dt of X plus delta X minus d by dt of X or we can write it as d by dt of delta X.

Put 3 into 2 $\Longrightarrow$ $\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla}\cdot\vec{U} = 0$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec{\nabla}\cdot\vec{U}$$

$$\boxed{\frac{D\rho}{Dt} = -\rho\left(\vec{\nabla}\cdot\vec{U}\right)}$$

Lagrangian form of Continuity equation

$$\boxed{\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{U})}$$

Eulerian control volume form of Continuity equation

The point is I am going to call this one as U delta U. So, we can now write 1 by delta V d by dt of delta V is already 1 by delta X d by dt of delta X plus 1 by delta Y d by dt of delta Y plus 1 by delta Z d by dt of delta Z which is nothing but d by dt of delta X. This is nothing but delta U. So, this is delta U by delta X plus delta V by delta Y plus delta W by delta Z. When you say limit delta X or delta Y tends to 0, you can write it as dou U by dou X plus dou V by dou Y plus dou W by dou Z.

All of it is simply the divergence of velocity, the three dimensional velocity. What is it that we have? This is 1 by delta V of d by dt of delta V is equal to the velocity divergence. So, we have two terms. One is here, this one carefully and another is this term. So, this term has come from, both of these terms have come out from this simple conservation where we expect the relative change of rate of change of mass is equal to 0.

Now, we have this term is equal to the velocity divergence and we have to think about this term 1 by rho d rho by dt. Let us write whatever we know 1 by rho d rho by dt plus del dot U is equal to 0. What have I done? Because all of this to be del dot U, I have simply substituted all of that into this equation. Let us say we call this equation as equation number 2 and this is the primary equation, equation number 1 and this is equation number 3. So, the idea is put equation number 3 into 2.

I will give you this or 1 by rho d rho over dt is equals to minus del dot U or capital D rho by dt is rho times minus we put minus outside del dot U. This equation is again the same, it is the continuity equation. The only difference is that in this approach we have let the control volume move along the fluid and we got the final expression in terms of total derivative. Earlier we had this expression which is this is the equation of continuity if it is Eulerian control volume. Using some vector identities we have converted that into total derivative but strikingly these 2 forms you see this capital D rho by dt is equals to minus rho times the velocity divergence we have got the same expression using the Lagrangian approach.

This is the form of Lagrangian form of the continuity equation. So, this is the basic idea of conservation of mass and how conservation of mass can be established using Eulerian as well as Lagrangian control volume and how these 2 different forms of continuity equation are actually one and the same. Let us just write this one, so dou rho by dou t is minus del dot rho U. So, this is the Eulerian control volume using the Eulerian control volume or Eulerian form of continuity equation. So, we will continue to explore more aspects of treating plasma as a fluid in the next lectures. Thank you.