Plasma Physics and Applications

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Week – 07

Lecture 32: Plasma as a Fluid: Fluid Equation -I

Hello dear students. In continuation to our discussion in the last class, we will consider plasma as a fluid and we will try to derive some equations which are the governing equations or the basic equations to understand plasma in a fluid state. In the last lecture, we have seen the meaning of a total derivative and how we can consider the motion of fluid or construct the change of a physical parameter while following the motion. So, this total derivative is written as this, let us say dT by dT. The rate of change of temperature with respect to the time will depend on the local change of temperature with time plus an additional term which is called as the advection term. Now, this u represents the velocity of the fluid and T as usual can be any dynamical variable which will or about which you are trying to understand.

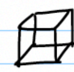
In the last lecture, we also made it clear that when you are considering fluid, all the equations of motion are to be written with respect to a small fluid volume which we call as the control volume. So, this control volume can be considered as the basic entity of the fluid. In a fluid, the microscopic behaviour at the level of each particle, each electron and ion is irrelevant. We only consider what is the combined behaviour of all these particles in a macroscopic point of view for example.

So, this control volume is the smallest element in the fluid which seems to be representing the entire fluid. So, this control volume can be of two types. Number one, a Eulerian control volume and number two, a Lagrangian control volume. So, when we will discuss the plasma behaviour as a fluid, we will write the complete set of equations which are sufficient to describe plasma. The most fundamental equation of those is the equation of continuity.

So, when you are talking about a fluid, you obtain all these governing equations by following the conservation. One we will look at the conservation of mass, then we will look at the conservation of momentum and we will look at the conservation of energy. These three different conservation will give us the entire governing equations. This is a very important topic in plasma physics. So, the Lagrangian derivative is this d by dt and the Eulerian derivative can be.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{U} \cdot \vec{\nabla} T$$

1) Eulerian $\frac{\partial}{\partial t}$



2) Lagrangian. $\frac{D}{Dt}$

Control Volume

1) Conservation of mass
2) Conservation of momentum
3) Conservation of energy.

$\Big\}$ Governing equations

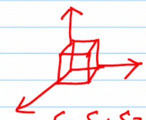
Continuity equation $<$ Lagrangian / Eulerian

Now let us talk about conservation of mass. The conservation of mass results in what is called as the continuity equation. We are familiar with the equation of continuity which enables us to understand how much of mass is coming into this control volume and how much of mass is going out of this control volume. So, for an incompressible fluid, this equation of continuity will complement the other equations for example momentum equation and all. So, this equation of continuity can be obtained for both Lagrangian control volume and as well as the Eulerian control volume.

So, both of these control volumes can be taken independently and we can obtain the equation of continuity. So, in order to do that let us consider first the first derivation is the equation of continuity and the approach that we are going to follow is using an Eulerian control volume. I must mention the basic difference between these two control volumes. The first one being a Eulerian control volume is a fixed three dimensional space let us say for example and this three dimensional volume is not going anywhere. It is fixed with respect to the coordinate system.

So, if you take this to be your coordinate system, this control volume is fixed with respect to the coordinate system. So, the fluid rather is going through this control volume. So, then for conserving mass you will have to look at the budget of the fluid that is entering into the volume and that is leaving from the volume. Ideally if these two quantities are equal then you would say that mass is conserved. There may be instances when the outgoing fluid volume or the amount of fluid can be smaller or larger in comparison to the incoming flux.

The equation of continuity (Eulerian control volume)



$\delta x \; \delta y \; \delta z$

$\delta V = \delta x \delta y \delta z$.

$$\rho u = \text{Mass flux} = \frac{\text{Rate of change of mass}}{\text{Area}} = \frac{\dot{m}}{A}$$

$$\dot{m} = \lim_{\delta t \to 0} \frac{\Delta m}{\Delta t} = \frac{M}{T}$$

$$\text{'}\rho\text{'} : \text{density of fluid} = \frac{M}{V}$$

$$\text{Mass flux} = \frac{MT^{-1}}{L^2} = \frac{\rho L^3 \times T^{-1}}{L^2} = \rho \underbrace{LT^{-1}} = \rho u$$

$$\boxed{\text{Mass flux} = \rho u}$$

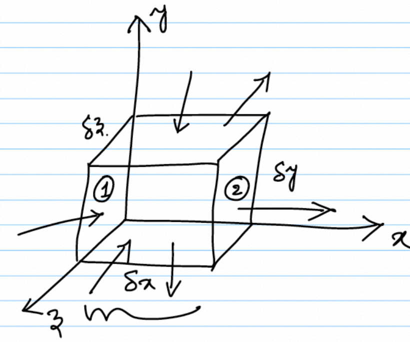
$$\boxed{\text{Mass flow rate} = \rho u A}$$

But your equation of continuity should be able to have a provision to accommodate all these three different possibilities. Coming back the Eulerian control volume is just a three dimensional space which is fixed with respect to the fluids movement or with respect to the coordinate system. What it means is that this is not going anywhere it is the fluid rather which is going through this control volume. So, that is the basic definition of an Eulerian control volume. So, since it is a fixed space we can assume it to be having sides delta x, delta y and delta z which means the volume of the fluid is delta v is delta x, delta y, delta z.

Now in order to proceed further we will define a parameter which is called as mass flux. We are after all trying to conserve the mass. So, what is mass flux? Mass flux is rate of change of mass divided by area. The rate of change of mass per unit area is called as the mass flux. So, we can write it as m dot divided by A.

So, m dot is basically can be written as lambda limit delta T tends to 0 delta m by delta T or we can simply write it as capital M by T. Now, if density of the fluid is rho, rho is what is rho is the density of the fluid is rho then mass flux can be written as you know that this is m T power minus 1 divided by L square. So, mass can be replaced as mass is nothing but sorry density is nothing but mass per unit volume. So, this can be written as mass is rho times L cube times T power minus 1 divided by L square. So, the mass flux is rho L T power minus 1.

$$\vec{U} = (u, v, w)$$
$$\downarrow \; \downarrow \; \downarrow$$
$$\delta x \;\; y \;\; z$$



Mass flow rate through the face 1

$$\dot{m} = \rho u \times \text{Area}$$

incoming $\longrightarrow$ $\dot{m} = \rho u \, \delta y \, \delta z$

$$f(x) + \frac{\partial f(x)}{\partial x}\,\underline{\delta x}$$

Mass flow rate at the surface 2

$$\rho u \, \delta y \, \delta z + \frac{\partial}{\partial x}\left[\rho u \, \delta y \, \delta z\right]\delta x$$

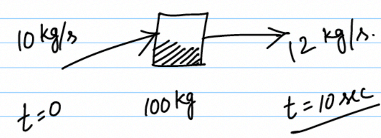
$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\,\delta x\right]\delta y \, \delta z \quad \Longleftarrow \text{Exiting}$$

So, this is the velocity of the fluid that can be written as rho times U. What is mass flux for the sake of understanding? Mass flux is the rate of change of mass occurring per unit area. Let us say you consider an area mass flux represents how much change in mass is happening per unit area. So, the mass flux can be written as rho U. Where we define this mass flux we will use this in the equation of continuity.

Now, since you have considered this volume we have to identify several other parameters. So, we will also define what is called as mass flow rate. Mass flow rate is the rate of change of mass. So, which will be mass flow rate would be rho U A. Mass flow rate is nothing but m dot the rate of change of mass.

So, from this definition m dot I am writing m dot. So, since mass flux I know as now I know as rho U. So, mass m dot can be written as rho U times the area, area is going to the other side. So, we have mass flux as rho U and mass flow rate as rho U A. That means, this within this volume we have the measure of this volume in the units of capital A or the area of the element.

Direction	Mass $\dot{m}$	out
x	$\rho u \, \delta y \, \delta z$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\delta x\right]\delta y \, \delta z.$
y	$\rho v \, \delta x \, \delta z$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\delta y\right]\delta x \, \delta z.$
z	$\rho w \, \delta x \, \delta y$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\delta z\right]\delta x \, \delta y.$



Now, let us consider the control volume and then try to use these concepts in order to understand or to derive the equation of continuity. So, for this purpose what we will do is we will take a three dimensional infinitesimally small volume and this is the volume that we have taken. So, this is x, this is y and this is z is coming out of the picture. So, this becomes delta x, this becomes delta x, this becomes delta y and this becomes delta z. So, the velocity of the fluid is in the components of U, V, W.

The U component is along delta x or x, V component is along y, the W component is along z. So, this control volume is now kept in the fluid and the fluid is going in and out of this control volume along the sides of this control volume. So, we call this side as 1 and this side as 2. Now the mass flow rate through the phase 1, mass flow rate is m dot, mass flow rate simply tells you what is the change in the mass per unit time, rate of change of mass. And what we have realized is the rate of change of mass is given as rho times the velocity times the area.

So, this area is there. So, only through this area how much mass is changing. So, if you have to write it mass flow rate is rho U the mass flux times area. So, how do you calculate the area of this side, side number 1. So, the sides are delta y and delta z. So, m dot becomes rho U delta y delta z.

$$\frac{dM}{dt} = in - out$$

$$= \rho u \,\delta y\, \delta z + \rho v\, \delta x\, \delta z + \rho w\, \delta x\, \delta y - \left[\rho u + \frac{\partial}{\partial x} \rho u\, \delta x\right] \delta y\, \delta z - \left[\rho v + \frac{\partial}{\partial y} (\rho v)\, \delta y\right] \delta x\, \delta z - \left[\rho w + \frac{\partial}{\partial z} \rho w\, \delta z\right] \delta x\, \delta y$$

$$= \cancel{\rho u\, \delta y\, \delta z} + \cancel{\rho v\, \delta x\, \delta y} + \cancel{\rho w\, \delta y\, \delta x} - \cancel{\rho u\, \delta y\, \delta z} - \frac{\partial}{\partial x} \rho u\, \delta x\, \delta y\, \delta z - \cancel{\rho v\, \delta x\, \delta y} - \frac{\partial}{\partial y} \rho v\, \delta x\, \delta y\, \delta z - \cancel{\rho w\, \delta x\, \delta y} - \frac{\partial}{\partial z} \rho w\, \delta x\, \delta y\, \delta z$$

$$\boxed{\frac{dM}{dt} = \left[-\frac{\partial}{\partial x} \rho u - \frac{\partial}{\partial y} \rho v - \frac{\partial}{\partial z} \rho w\right] \delta x\, \delta y\, \delta z}$$

What is this? This is the mass flow rate through the phase A. So, that means this much amount of mass is changing per unit time along the phase 1. So, where is this mass going? This mass is let us say can be considered to be going inside. Now, is it possible for us to here we know how much mass is going in. So, can we also find out how much mass is changing from the phase 2 and if these two masses are same or the rate at which the mass is changing on this area is same, we can reasonably say that mass is conserved, but that is only in the one direction.

We will have to consider the other directions as well just to be able to say mass is conserved. So, how do we calculate this mass flux or mass flow rate through delta y? Because now you can write a simple expression, but what you have to keep in mind is let us say you assume this much of this much amount of mass to be entering. So, with respect to this quantity, how do you write the mass flow rate through the phase number 2? So, that is simple. Let us say if you know the value of the function at a point along the axis as fx and if you are asked to find out what will be the functions value at a later point on the same axis, then what you will do is you will take a gradient that will be dou fx by dou x and multiply with the increment of the distance that you have travelled. So, similarly this is the distance that you are referring to because the phase 2 is situated at delta x away from this point.

So, now keeping that standard notation, we can write mass flow rate at the surface 2 times rho u delta y delta z plus dou by dou x of rho u delta y delta z times delta x or we can write it as rho u plus dou by dou x of rho u delta x times delta y delta z. So, you understand what I have done. So, I have taken delta y delta z as common and kept it outside the bracket. So, what is this? We can assume this to be the incoming flux into the control volume and this can be assumed to be the exiting flux out of the Eulerian control volume. But it is only along one direction which is it the x direction.

So, that means that you have to be able to write the flux going in and coming out from all the different phases and then you have to sum all of them and then comment on how or what are the parameters on which the rate of change of mass will actually depend on. So, we will write the direction. So, this is mass flow rate in and out coming in and going out. So, direction is along the x direction which we have done just now which is rho u delta y delta z and this will be rho u plus dou by dou x of rho u times delta x multiplied by delta y delta z. Always remember delta y delta z is the perpendicular area element along the direction that you have considered.

So, we have y which is rho v delta x delta z and it will be rho v plus dou by dou y of rho v delta y delta x delta z and we have the third which is z rho w delta x delta y and this will become rho w plus dou by dou z of rho w delta z delta x delta y. Now, ideally the net mass that is going in or the flux that is going in should be equal to the net that is going out. So, if you have for example a small tank and if the water is going in at a rate of let us say 10 kg per second and if it is coming out at a rate of 2 kg per second and if you start your time at let us say t is equal to 0 seconds and t is equal to 10 seconds. What would happen? At t is equal to 0 seconds there is no mass inside as time goes on. So, within a span of 10 seconds you would have accumulated a mass of 100 kg inside.

But at the same time during this 10 seconds 2 kgs of mass is going out. That means 2 kgs of mass is going out. So, 20 kgs of mass has already exited. So, now you have remaining 80 kgs which is accumulated inside. So, what do you comment on this situation? You say that the net mass that is going out is not same.

So, there is a sink inside the system. So, mass is getting piled up inside or let us say if you have 12 kgs per second which is the exit then what would happen? So, you are getting more mass out rather than the amount of mass that is going in. So, in this case you would expect a source inside the system. So, this is the difference. This is what we have to comment on. What it means is now we have to account for all that is going in and subtract all that is going out then you will get the rate of change of mass.

So, ideally the rate of change of mass dM by dT should be what is going in minus what is going out. So, we will just write all of these things together which will be rho u delta y delta z plus rho v delta x delta z plus rho w delta x delta y minus rho u plus dou by dou x of rho u delta x times delta y delta z minus rho v plus dou by dou y of rho v delta y delta x delta z minus rho w plus dou by dou z of rho w delta z times delta x delta y or rho u delta y delta z plus rho v delta x delta y minus rho u. I am just expanding all the terms so that the algebra can easily be understood minus dou by dou y of rho v delta x delta y delta z minus rho w delta x delta y minus this is what we have all the terms. Now we can see that some terms will get cancelled. So, what are we left with? We are left with the remaining terms.

We will gather all those terms that is the sum of those is equal to the rate of change of mass m by dt the rate of change of mass that should be minus dou by dou x of rho u minus dou by dou y of rho v minus dou by dou z of rho w multiplied by delta x delta y delta z. So, we will stop at this step. We will take this ahead and write this expression in terms of parameters that we know so that we can get a simplified equation of continuity. We will continue this in the next lecture. Thank you.