Plasma Physics and Applications

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Lecture 29: Motion in Time Varying Electric field

Hello dear students. We are trying to understand the particle motion in non uniform magnetic fields. So, particle motion means we are trying to understand plasma when it is subjected to non uniform magnetic field. So, in this class we will try to see how a time dependent electric field will change the particles trajectory or what additional information can we learn about the motion of a particle in a non uniform or time varying electric field. So, we will consider a slowly varying time dependent electric field which means the rate of change of electric field with respect to the time is not 0. So, for this case we will try to have a configuration in which the electric field and the magnetic field are perpendicular to each other.

So, which means E of t dot B is equal to 0. So, the dot product is cos theta cos 90 is equal to 0. So, we have a configuration like this. Now we shall now investigate the motion of charged particles in a time dependent electric field.

So, where do we start? We always start with the basic Lorentz force equation which is m dV by dt is Q times E plus V cross B. So, let us say we call this equation as equation number 1. Now whether the electric field is time dependent or not the effect of electric field and magnetic field is on the velocity of the particle. Now the velocity of the particle can be written as the parallel component of the velocity and the perpendicular component of the velocity. Now we know this parallel component is defined in such a way that it is parallel to the magnetic field the direction of magnetic field.

So, we can now decompose this equation number 1 into 2 equations that is one equation representing the motion along the parallel component another equation representing the motion along the perpendicular component. So, we have 2 equations now m dV perpendicular by dt is equal to Q times E perpendicular plus V perpendicular cross B. Let us say we call this equation as 2. So, perpendicular component still has this non-zero curl V cross B is still non-zero and we have the other equation the decomposed equation dV parallel by dt is equal to Q times E parallel. Let us say we have this as equation number 3.

Slowly varying time dependent Electric field.

$$m\frac{d\vec{V}}{dt} = q(\vec{E} + \vec{V}\times\vec{B}) \quad \text{—— (1)}$$

$$V = \underline{\underline{V_{\parallel}}} + V_{\perp}$$

$$m\frac{dV_{\perp}}{dt} = q(\underline{E_{\perp}} + V_{\perp}\times B) \quad \text{—— (2)}$$

$$m\frac{dV_{\parallel}}{dt} = qE_{\parallel} \quad \text{—— (3)}$$

$$\frac{d\vec{E}}{dt} \neq 0$$

$$\vec{E} \perp \vec{B}$$

$$\vec{E}(t)\cdot\vec{B} = 0$$

(3) denotes acceleration along $\vec{B}$

Divide 'q' & multiply $\frac{\vec{B}}{B^2}$

$$\frac{m}{q}\frac{d\vec{V}_{\perp}}{dt}\times\frac{\vec{B}}{B^2} = \frac{\vec{E}_{\perp}\times\vec{B}}{B^2} + (\vec{V}_{\perp}\times\vec{B})\times\frac{\vec{B}}{B^2}$$

$$(\vec{A}\times\vec{B})\times\vec{C}$$

What happened to the curl of velocity with the magnetic field? Because these two are parallel to each other sin theta is 0 that is why we do not have the other term appearing. So, what is it that we have done? We have written one Lorentz force equation and we have decomposed this equation into two equations one representing velocities change along perpendicular another along the parallel directions. Now, the third one denotes acceleration along the magnetic field. The equation number 3 denotes acceleration along the magnetic field. Now, let us say we perform some simple algebra onto these equations and say divide equation number 3 with Q and multiply with B by B square multiply or take a cross product from the right.

Then what we have is M by Q we are dividing with Q dV perpendicular by dt cross B by B square is equal to now Q is no longer there on the right hand side. So, we have E perpendicular cross B by B square plus V perpendicular cross B cross B by B square. We have just brought the Q onto the left hand side and multiplied the entire equation with B by B square. Why are we doing this? We are doing this just to write this equation in terms of things or terms that we are familiar with. Now, what you see on the right hand side is you are seeing this E cross B drift velocity right away and you are seeing a triple vector product.

$$\frac{m}{q}\frac{d\vec{v}_\perp}{dt}\times\frac{\vec{B}}{B^2} = \frac{\vec{E}_\perp\times\vec{B}}{B^2} - \frac{\vec{v}_\perp(\vec{B}\cdot\vec{B})}{B^2} + \cancelto{0}{\frac{\vec{B}(\vec{v}_\perp\cdot\vec{B})}{B^2}}$$

$$\vec{v}_\perp\cdot\vec{B} = 0$$

$$\frac{m}{q}\frac{d\vec{v}_\perp}{dt}\times\frac{\vec{B}}{B^2} = \frac{\vec{E}_\perp\times\vec{B}}{B^2} - \frac{\vec{v}_\perp B^2}{B^2}$$

$$\frac{m}{q}\frac{dv_\perp}{dt}\times\frac{\vec{B}}{B^2} = \frac{\vec{E}\times\vec{B}}{B^2} - \vec{v}_\perp \quad \text{--- (4)}$$

$$\vec{v}_\perp = \vec{v}_{ac} + \vec{v}_E + \vec{v}_p \quad \text{--- (5)}$$

$\vec{v}_{ac}$: The gyration

$\vec{v}_E$: $\vec{E}\times\vec{B}$ drift velocity

$\vec{v}_p$: Polarization drift

So, let us resolve this triple vector product and see what are the terms that we get. So, we will have M by Q dV perpendicular by dt cross B by B square is equal to E perpendicular cross B by B square minus V perpendicular times B dot B divided by B square plus B times B square. What have we done? We have done A cross B cross C. This is in the form of A cross B cross C. Using the standard vector formula, we have expanded this onto these two terms.

Now we know that V perpendicular is defined such that it is making right angles to the magnetic field. So, this dot product will be 0 which means this term will be 0 and in effect what we have is we have M by Q dV perpendicular by dt cross B by B square is equals to E cross B by B square minus V perpendicular times B square by B square or M by Q. Let us say we call this equation as 4. What do we have? We have done nothing. We have removed this term and B square gets cancelled and we have a slightly modified Lorentz force equation which is having the E cross B drift velocity on the right hand side.

So, solution of this sort of differential equation can be assumed to be a sum of a time dependent part and a constant. So now, the solution is V perpendicular. Of course, V perpendicular is a solution because we are only looking at the decomposing equation equation number 2 not equation number 3. So, the solution of this equation is V perpendicular. So, V perpendicular can be written as V AC plus VE plus V P.

$$\frac{m}{q}\frac{d}{dt}\left[\vec{v}_{ac}+\vec{v}_E+\vec{v}_P\right]\times\frac{\vec{B}}{B^2} = \frac{\vec{E}_\perp\times\vec{B}}{B^2} - \vec{v}_{ac} - \vec{v}_E - \vec{v}_P$$

$$\frac{m}{q}\left[\frac{d\vec{v}_{ac}}{dt}\times\frac{\vec{B}}{B^2} + \frac{d\vec{v}_E}{dt}\times\frac{\vec{B}}{B^2} + \frac{d\vec{v}_P}{dt}\times\frac{\vec{B}}{B^2}\right] = \cancel{\frac{\vec{E}_\perp\times\vec{B}}{B^2}} - \vec{v}_{ac} - \cancel{\vec{v}_E} - \vec{v}_P$$

$$\frac{m}{qB^2}\left[\frac{d\vec{v}_{ac}}{dt}\times\vec{B} + \frac{d\vec{v}_E}{dt}\times\vec{B} + \frac{d\vec{v}_P}{dt}\times\vec{B}\right] = -\vec{v}_{ac} - \vec{v}_P$$

$$\frac{d}{dt}V_E = \frac{E\times B}{B^2} = 0$$

$$V_E = \frac{E(t)\times B}{B^2}$$

$$\dot{v}_E = \frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right) \neq 0$$

$$m\vec{a} = \vec{F}$$

$$a = \frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right)$$

$$a = \frac{d}{dt}\left(\frac{\vec{E}(t)\times\vec{B}}{B^2}\right)$$

What is the meaning of this? One we have to remember that if we have this sort of differential equation, the solution can be decomposed into a time dependent part and a constant part. So, what I have done is I have written three terms not just two terms. What are these three terms? V AC. What is V AC? This represents the gyration. The gyration of the particle because of the presence of magnetic field is represented by V AC.

And VE. What is this? This is the E cross B drift velocity. This is the E cross B drift velocity we are familiar with. Whenever we have electric and magnetic fields perpendicular to each other, we get a velocity which is called as the E cross B drift velocity which should be there. The electric field is already there. So, it should be there.

Now, we have to think of something else which should also be accommodated in the same picture. What is the something? The electric field is changing with respect to time. That means that there is some acceleration because of the change of electric field with respect to time. So, this change which is brought in just because of time varying electric field is VP. This is the additional component which comes into picture just because of the electric field being time dependent.

This is called as the polarization drift. VP is referred to as polarization drift. Now, these two are very familiar to us. Whenever we have electric field, the particles movement we know V parallel remains constant along the direction of magnetic field because the equation number probably 3 already tells you that. The perpendicular component is what makes V perpendicular square is equal to square root of Vx square plus Vy square.

So, Vx square plus Vy square put together is executing the circular motion or the gyration. The second thing is this we need to have this constant E cross B by B square. This is the constant drift velocity. Now, the moment electric field is no longer constant, it will contribute to another drift velocity which is called as the polarization drift velocity. Now, let us say we call this decomposition of velocity as equation number 5 and we have to substitute equation number 5 into equation number 4.

$$\vec{F} = ma = m \frac{d}{dt}\left(\frac{\vec{E} \times \vec{B}}{B^2}\right) = m \frac{\dot{E} \times B}{B^2}$$

$\vec{F}$ is a direct consequence of time dependent Electric field

$$\vec{v} = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

$$\vec{v}_p = \frac{1}{q} \frac{\left[m \frac{\dot{\vec{E}} \times \vec{B}}{B^2}\right] \times \vec{B}}{B^2}$$

$$= \frac{m}{qB^2} \frac{\dot{\vec{E}} \times \vec{B}}{B^2} \times \vec{B}$$

$$\vec{V}_p = \frac{m}{qB^2} \frac{d}{dt}\left[\frac{\vec{E} \times \vec{B}}{B^2} \times \vec{B}\right]$$

Now, when we do that, what we have is a bit lengthy equation which is m by Q T by dT of it should be substitute on the both sides. V AC plus VE plus VP cross B by B square is equals to E perpendicular. It is to call E cross B by B square minus V AC minus VE minus VP. This is what I forgot to put symbols on these velocities. Now, we will take this B by B square inside and we will be able to write m by Q dV AC by dT cross B by B square plus dVE by dT cross B by B square plus dVP by dT cross B by B square is equals to E perpendicular cross B by B square minus V AC minus VE minus VP.

Now, we know that the E cross B drift velocity is equal to E cross B by B square. These two things will get cancelled because both of them are one and the same. Now, when the electric field is now changing with respect to time, we have got a additional term which is called as the polarization drift just because of the variation of electric field with respect to time. Now, what we have is we have all of this m by m by QB square times dV AC by dT cross B plus dVE by dT cross B plus dVP by dT cross B is equals to minus V AC minus VP. Now, the electric field E is responsible for giving this E cross B drift velocity.

Now, as long as this electric field is constant with respect to time, the drift velocity will be a constant. But when the electric field is changing with respect to time, this drift velocity should also change. account that particular variation is a very important thing. Now, we have two additional terms. So, VD is the now since electric field is time dependent, so the particles trajectory should also change with respect to time.

So, we can say that Ve dot is d by dT of E cross B by d square. Let us try to understand what I have written. So generally, Ve is E cross B by d square. Before we discuss this last formula, let us go back and just try to comprehend what we have done so far. So, we are considering a slowly varying time dependent electric field.

We have started with the basic Lorentz force equation. We have decomposed this Lorentz force equation into two equations which are like one for the perpendicular component and therefore, the parallel component. And we performed a simple mathematical operation of dividing this equation by the charge and multiplying with B by B square. And once we do that, we have the other terms that appear on the right hand side. And since the perpendicular component seems to be affected by all of this on the right hand side.

The perpendicular component of velocity seems to be affected by this one, seems to be affected by all of these things which appear on the right hand side. That means we have to see how these things will affect. So when we do this mathematical operation of expanding the triple vector product, we get all these terms. And since the perpendicular component is at right angles to the magnetic field, we can make this term to be 0. Now since the perpendicular is affected by three things actually.

One an electric field, two a magnetic field and three the time dependence of the electric field. So, these three things are combinedly written in these three velocities. One represents the gyration of the particle around the guiding center and second one is the E cross B drift velocity which is just a combination of E and B. And third one is an additional component of velocity which comes into picture just because the electric field being time dependent. Then we did some simple algebra and removed all the terms which are equal to each other on the right hand side.

Now coming back, if the electric field is not time dependent, when you write this and take a derivative of this with respect to time, what will happen? Since these are not time dependent, you will simply get a 0. But if the electric field is time dependent, then you will have Ve dot becoming non-zero. So what is Ve dot? Ve dot is acceleration and if you multiply with m, you will get the force that is there because of this Ve dot. So how do you write the acceleration? The acceleration A can be written as d by dt of E cross B by B square. The rate of change of velocity and that velocity is nothing but or it is more appropriate to write the acceleration as d by dt of E of t cross B by B square.

Then we will use the formula that we know, F is equal to ma. So the force has to be m times d by dt of E cross B by B square. So this force F is a direct consequence of the time dependent field or it is more appropriate to say that this force is a direct consequence of electric field being time dependent. So this drift is called as the polarization drift. If we have to get the expression for Vp or some velocity component which is because of a generalized force or if I write the expression in terms of just V or we know that this is equal to 1 by Q F cross B by B square.

So we have to now substitute this F, this relation for F into this. So this velocity is called as the polarization drift velocity which is equal to 1 by Q times ma dot cross B by B square cross B divided by B square. You understand what I did? I took this expression, I took the time derivative inside then I can write m E dot cross B by B square. So all of this is the force, 1 by Q F cross B by B square or we will now simplify this to make it look simple or easy to remember m by QB square times E dot cross B by B square cross B divided by B square or since we do not know how E dot will curl with B, it is okay to write this time dependence outside and write m by QB square times d by dt of E cross B by B square. There should be only one B square here, so this is already there.

E cross B by B square cross B. So this is the polarization drift. So we will stop here and we will take this discussion ahead in the next lecture. Thank you.