Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week – 06

Lecture 26: Magnetic Mirroring - IV

Hello dear students. So far we have understood what is magnetic mirroring, how we can confine plasma using a bottle arrangement in which the magnetic field is very strong at the extremities and in between it is weak. So, what this arrangement does is it will trap plasma between these two points. So, this is a very important topic in plasma physics how we trap plasma between two points using magnetic field. So, what we do in today's class is we will discuss numericals based on magnetic mirroring. So, this is this numericals will help you understand the concept much better.

the course evaluation is primarily going to be numerical it will help if you solve as many examples or as many numerical problems as possible. So, this is numerical problems on magnetic mirroring. So, let us say the first problem is let us consider a magnetic mirror configuration such that the actual magnetic field along the axis is given by this relation. What does this expression mean? The magnetic field as a function of Z varies like this V0 times 1 plus alpha Z square.

Numerical Problems on Magnetic mirroring

1) $B(z) = B_0(1+\alpha z^2)$ $\longrightarrow \hat{z}$

- Calculate the mirror points (Reflection points) for an e^- with $\underline{v}$ initially at $z=0$

Initial pitch angle $\underline{\alpha_0}$

$v_{\perp_0} = v\sin\alpha_0$

$v_{\parallel_0} = v\cos\alpha_0$ $\qquad v_\parallel = 0$

$\rightarrow W_0 = W_m$ & $\mu_0 = \mu_m \leftarrow$

$$\cancel{\tfrac{1}{2}}\cancel{m}(v_{\perp_0}^2 + v_{\parallel_0}^2) = \cancel{\tfrac{1}{2}}\cancel{m}(v_{\perp_m}^2 + \cancel{v_{\parallel_m}^2}^{\,0})$$

$$\boxed{v_{\perp_0}^2 + v_{\parallel_0}^2 = v_{\perp_m}^2} \quad \text{— (a)}$$

Z is just a constant alpha is a constant and Z is the coordinate which is let us say like this. Now what we have to do is if this is the configuration that we have been given with now we have to calculate the mirror point calculate the mirror points. What are the mirror points? Where the reflection will happen? V-section points for an electron with a velocity V initially at Z is equal to 0. So, the electron is imagined having a velocity V which is present at Z is equal to 0. So, when Z is equal to 0 as the particle travels away from Z is equal to 0 it starts experiencing this type of magnetic field and also assume that the initial pitch angle is alpha 0.

So, this is the information that we have been given with now we have to calculate where if this is in this configuration where exactly the particle will mirror or reflect back. So, when the particle is reflecting back the only thing that we have to understand at that point the perpendicular component velocity is the only surviving part of the velocity the parallel component of the velocity becomes simply 0. So, we have the angle the pitch angle which is just a measure of how these two velocities are separated especially. So, we can write the perpendicular initial velocity with the help of this angle as V sin alpha 0 perpendicular initial velocity and the parallel velocity as just the component of velocity with this pitch angle. So, V perpendicular is equal to V sin theta V parallel is equal to V sin V cos theta we know how to draw components simple vector analysis.

$$\mu_0 = \mu_m$$

$$\underbrace{\frac{m v_{\perp_0}^2}{2 B_0}}_{\text{Initial}} = \underbrace{\frac{m v_{\perp_m}^2}{2 B}}_{\text{Mirroring point}}$$

$$B = B_0(1+\alpha z^2)$$

$$B(z) = B_0(1+\alpha z^2)$$

$$z = 0 \qquad v, \alpha_0$$

$$\mu = \frac{W_\perp}{B} = \frac{m v_\perp^2}{2B}$$

$$B_0(z) = \underline{B_0}$$

$$\frac{\cancel{m} v_\perp^2}{\cancel{2 B_0}} = \frac{\cancel{m} v_{\perp_m}^2}{\cancel{2} \cancel{B_0}(1+\alpha z_m^2)}$$

$$v_{\perp_0}^2(1+\alpha z_m^2) = v_{\perp_m}^2$$

Using (a)

$$v_{\perp_0}^2(1+\alpha z_m^2) = \underbrace{v_{\perp_0}^2 + v_{\parallel_0}^2}$$

$$\cancel{v_{\perp_0}^2} + v_{\perp_0}^2 \alpha z_m^2 = \cancel{v_{\perp_0}^2} + v_{\parallel_0}^2$$

So, at the initial point let us say the initial point is indicated with a subscript of 0 let us say at the initial point the energy the kinetic energy is WM and the magnetic moment mu 0 is equal to mu m. Let us say we write the conservation which is half m V perpendicular 0 square plus V parallel 0 square which has to be equal at the mirroring point which is V perpendicular m square plus V parallel m square. The suffix m indicates the mirroring point and 0 indicates the initial point. So, between these two points the energy has to be conserved what goes in between is actually a matter of interest. But we know that as the so this all things can be cancelled as the particle is about to get reflected the parallel component will not survive.

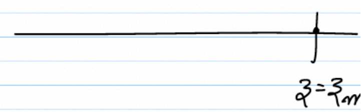
So, V parallel has to be 0 so at the mirroring point this velocity component V parallel m has to be 0 so this part can be straightaway neglected. So, why the velocity initially is not 0 in the parallel direction because the particle is just experiencing in normal magnetic field which is not sufficient enough to reduce the V parallel to 0 to keep the magnetic moment constant. What it means this one is that the kinetic energy is the same between both the points between the initial point and the mirroring point so does the magnetic moment initially and at the mirroring point both of them are same. So, we can rewrite this expression as V perpendicular 0 square plus V parallel 0 square is equals to V perpendicular m square. This is not something new this is we know it very well.

$$v_{\perp_0}^2 \alpha z^2 = v_{\parallel_0}^2$$

$$B = B_0(1+\alpha z^2)$$

α_0, β

@ Mirroring point $z = z_m$



$$v_{\perp_0}^2 \alpha z_m^2 = v_{\parallel_0}^2$$

$$\boxed{z_m = \frac{v_{\parallel_0}}{v_{\perp_0}\sqrt{\alpha}}}$$

$$\boxed{z_m = \frac{1}{\tan\alpha_0 \sqrt{\alpha}}}$$

$$\boxed{B(z) = B_0(1+\beta z^2)} \Leftarrow \boxed{z_m = \frac{1}{\tan\alpha_0 \sqrt{\beta}}}$$

So, initially there are the velocity parallel and perpendicular will exist but at the mirroring point only perpendicular component will exist. What is important for you to understand is this equation is valid just at the mirroring point on the right hand side and this one on the left hand side is valid everywhere but the mirroring point. This is using the conservation of energy let us also use the conservation of magnetic moment mu naught is equal to mu m initial magnetic moment is equal to the magnetic moment at the mirroring point. So, using things that we know m V perpendicular 0 square by 2 B naught because the magnetic moment is only dependent on the perpendicular velocity. So, we have written mu as W perpendicular by B or m V perpendicular square by 2 B.

So, there is no point of even discussing about the parallel component of velocity here which will be m V perpendicular m square. So, do not be confused with the m that appears in the suffix it is just the mirroring point this mass m is rather the mass by 2 B. So, now we have the magnetic field B naught. Now let us say we cannot write B naught here this is the initial condition this is initial and this one is at the mirroring point. So, what we are supposed to write is m V perpendicular square by 2 B naught.

Q2)

$$B(z) = B_0\left[1+\left(\frac{z}{a}\right)^2\right]$$

B_0 @ $t=0$

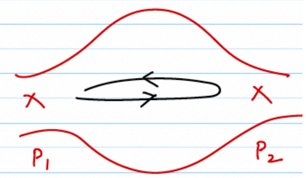
a is constant

$z = \pm z_m$

(i) Obtain z-component of velocity

(ii) Average force acting on the particle along z-axis

(iii) Find time period of S.H.M.



$$\mu = \frac{m v_\perp^2}{2B} = \frac{W_\perp}{B}$$

$$v_{\perp z}^2 = \frac{2\mu}{m} B(z)$$

$$v_{\perp z}^2 = \frac{2\mu}{m} B_0\left[1+\left(\frac{z}{a}\right)^2\right]$$

Now B is written as B naught times 1 plus alpha V square B which is a function of z B naught 1 plus alpha V square. Now according to the stem of the equation when z is 0 the particle has a velocity V at a pitch angle of alpha naught. So, when you substitute z is equal to 0 into this the initial magnetic field at the beginning is nothing but just B naught. So, B naught m V perpendicular square by 2 B naught is equal to m V perpendicular m square here the magnetic field is not just B naught it has to be 2 B naught times 1 plus alpha z square. Now let us say at the mirroring point let us denote the position coordinate of the mirroring point along the z axis as zm.

So, this B naught gets cancelled mass also gets cancelled. So, we have V perpendicular 0 times 1 plus alpha zm square is V perpendicular m square. So, from this earlier expression which is obtained using the conservation of energy the perpendicular component of velocity at the mirroring point square is can be written as this. So, which will be V perpendicular 0 square times 1 plus alpha zm square is equals to V perpendicular 0 square plus V parallel 0 square. If you are wondering how did I get this? This is using let us say we name this expression as A and we write here as using A.

So, this upon expansion V perpendicular 0 square plus V perpendicular 0 square alpha zm square is equals to this gets cancelled. And as a result what we have is V perpendicular 0 square alpha z square is V parallel 0 square. So, we have everything in terms of the initial vel initial velocities nothing which is representing the state of the electron at the mirroring point. What do we wanted? At the mirroring point at the mirroring point z is equals to zm. So, V perpendicular 0 square alpha zm square is V parallel 0 square.

$$\text{Energy} = \frac{1}{2} m v_{\perp z}^2 + \frac{1}{2} m v_{\parallel z}^2$$

$$@\ z = z_m \ (\pm)$$

$$KE = \frac{1}{2} m v_{\perp z_m}^2$$

$$\left.\rightarrow \mu = \frac{\frac{1}{2} m v_{\perp z_m}^2}{B_{z_m}} \right\} (a)$$

$$B_{z_m} = B_0\left[1 + \left(\frac{z_m}{a}\right)^2\right]$$

$$v_{\perp z_m}^2 = \frac{2\mu}{m} B_{z_m}$$

$$(b) \left\{ \rightarrow v_{\perp z_m}^2 = \frac{2\mu}{m} B_0 \left[1 + \left(\frac{z_m}{a}\right)^2\right] \right. \checkmark$$

So, zm can be written as V parallel 0 by V perpendicular 0 times square root of alpha. So, this is a very important result. So, zm is the point at which the magnetic field will make the particle bounce back into the weaker magnetic field and it so happens that this point along the axis of the magnetic field. This is z is equals to zm. This distance mainly depends on the velocity or the ratio of initial velocities but nothing else other than that.

So, using the definition of pitch angle we can write zm is equals to 1 by tan alpha 0 into square root of alpha. Now here what you have to understand is alpha 0 is the pitch angle and alpha is just a constant. Alpha has no relation whatsoever to the rest of the variables. Alpha simply denotes the speed with which the magnetic field is changing or increasing along the z axis because we have defined the magnetic field B as B naught times 1 plus alpha z square. So, it would have been convenient if we have written the pitch angle as alpha naught and this constant as beta.

$$\frac{1}{2} m v_{\perp_z}^2 + \frac{1}{2} m v_{\parallel_z}^2 = \frac{1}{2} m v_{\perp_{z_m}}^2$$

$$\Rightarrow v_{\parallel_z}^2 = \underline{\underline{v_{\perp_{z_m}}^2}} - v_{\perp_z}^2$$

$$v_{\parallel_z}^2 = \underbrace{\frac{2\mu B_0}{m}\left[1+\left(\frac{z_m}{a}\right)^2\right]}_{\text{Using (b)}} - \underbrace{\frac{2\mu B_0}{m}\left[1+\left(\frac{z}{a}\right)^2\right]}_{\text{Using (a)}}$$

$$v_{\parallel_z}^2 = \frac{2\mu B_0}{m}\left[\left(\frac{z_m}{a}\right)^2 - \left(\frac{z}{a}\right)^2\right]$$

$$v_{\parallel_z} = \sqrt{\frac{2\mu B_0}{m}\left(\left[\frac{z_m}{a}\right]^2 - \left[\frac{z}{a}\right]^2\right)}$$

Just so that if you are not following or to avoid confusion I can write it as tan alpha naught into square root of beta. This is the mirroring point. Now beta is just a constant. So, this formula will be valid if the magnetic field Bz is written as B naught times 1 plus beta z square as simple as that. But what is the concept that we have learned by solving this example is that the moment the particle picks up from z is equals to 0 the velocities of the parallel and perpendicular component of the velocity of the particle their magnitude and the angle at which the particle will shoot into the magnetic field will decide whether the particle is going to be trapped or not at the same time.

We have learned this in the loss cone but what this conveys is that not only the particles velocity the pitch angle is also important but the distance at which this particle can reverse back into the weak magnetic field is also decided by the ratio of the velocities and the angle at which the particle has started from the initial point. So, this is the most important conclusion about the magnetic mirroring. Not only we solved how the magnetic mirroring works the mathematics of it the underlying physics of the mathematical steps we have also been able to realize for any given type of magnetic field which has a spatial inhomogeneity we can find out where the point can be established for the mirroring to happen. Now this point the significance of this point lying within the bottle or outside the bottle is the topic of discussion for another class but with this understanding we can simply derive the relevant expressions. Let us just go back once just so that you are thorough with the process what did I do? The two conservation laws that I have to keep in mind while solving this example is that or not even example while the entire process of magnetic mirroring entirely relies heavily on the conservation of energy and the conservation of magnetic moment.

(ii)

$$B_z = B_0\left[1+\left(\frac{z}{a}\right)^2\right]$$

$$\frac{\partial B_z}{\partial z} = \frac{\partial B_0}{\partial z}\left[1+\left(\frac{z}{a}\right)^2\right]$$

$$\boxed{\frac{\partial B_z}{\partial z} = \frac{2B_0}{a^2}\,z} \quad \text{— (c)}$$

$$F_z = -\mu\frac{\partial B}{\partial z}$$

Using (c)

$$\boxed{F_z = -\mu\left(\frac{2B_0}{a^2}\right)z}$$

I have used both of them obtained a relation for the velocities used that particular relation for velocities the in the subsequent part where I have used the conservation of magnetic moment. So, this is important conclusion that we have we can derive about the magnetic mirroring. Let us take one more configuration of magnetic field this is even a more involved example which may take some time for you to solve. So, what I expect is I expect you to follow this procedure and try to understand how any problem can be solved when you have been given the magnetic field configuration. So, this is going to be question number 2.

Now the magnetic field configuration is B of Z is equals to B0 times 1 plus Z by A whole square. So, where B0 what is B0? B0 is the constant magnetic field at T is equals to 0 and A is also a constant and the mirroring will happen at Z is equals to plus minus Zm. Now this is the configuration the bottle configuration. So, you have this the magnetic field is like this is the bottle configuration. So, what are these cross points P1, P2 these are the points of confinement or the mirroring points.

$$m\frac{d^2 v_z}{dt^2} = q(\vec{v}\times\vec{B})$$

$$F_z = -\frac{2\mu B_0}{a^2} z.$$

$$m\frac{d^2 z}{dt^2} = -\frac{2\mu B_0}{a^2} z.$$

$$\frac{d^2 z}{dt^2} = -\frac{2\mu B_0}{ma^2} z.$$

$$\frac{d^2 z}{dt^2} + \underbrace{\frac{2\mu B_0}{ma^2}}_{\omega^2} z = 0$$

$$\boxed{\frac{d^2 z}{dt^2} + \omega^2 z = 0}$$

$$\omega^2 = \frac{2\mu B_0}{ma^2}$$

ω

This is given when you have been given this type of configuration what you are expected to find out or solve for is let us imagine a particle that is trapped in this particular magnetic field. We have to obtain the Z component of velocity. Then we have to obtain an expression for average force acting on the particle along the Z axis. Then if the particle is of course trapped it is going back and forth it goes to the mirroring point its perpendicular component velocity becomes maximum it bounces it goes into the weak field because now the particles velocity is in the negative Z direction it goes again in the negative Z direction it similarly faces a mirroring point it bounces back. So, this is a periodic motion or let us say simple harmonic motion.

So, find out the time period of the simple harmonic motion which is to be seen between the 2 mirroring points. So, this is the problem. Now let us see now we are asked a different set of questions. In the last example we have seen that we have to just get the points where the mirroring will happen. Now here we have to obtain a different set of parameters.

So, we will go step by step. So, we can define the magnetic moment we can write the magnetic moment Mu is m V perpendicular square by 2B or W perpendicular by B. So, from this V perpendicular along Z square is 2 Mu by m times P of Z and using the configuration that we are having here we can write 2 Mu by m times P naught into 1 plus. Now the total energy is half m V perpendicular square plus half m V parallel square. Let us define the mirroring points at Z is equal to Zm it is a plus minus either sides. So, at Z is equal to Zm the mirroring points the particle will be reflected back into the region of weak magnetic field and the rest of the story is very well known to you.

So, at the mirroring points what happens the kinetic energy is half m V perpendicular Zm square and the dipole moment Mu at that point is half m V perpendicular Zm square divided by B Zm because at the mirroring point. Now what is B Zm we know what is B Z. So, we will substitute Z is equal to Zm into that expression then we have the magnitude of the magnetic field at the mirroring point. I hope you are following I have not done anything other than just pulling the basic conservation energy of energy and writing the magnetic moment at the mirroring point not anywhere else.

So, that is what I have done. If you have done that let us see so from the first formula V perpendicular Zm square is equals to 2 Mu 2 Mu by m B Zm or V perpendicular Zm square is 2 Mu by m times B naught to 1 plus. Same thing actually once you are writing the magnetic moment at the mirroring point and using the same formula you are writing the perpendicular component of velocity at the mirroring point. So, this will be maximum this velocity will be maximum so is this magnetic moment. We are going to use both the expressions. Now for the energy of the particle to be conserved we will write half m V perpendicular Z square plus half m V parallel Z square is equals to half m V perpendicular Zm square because the parallel component at Zm will be 0.

So, we can cancel the half m that appears on all the sides and we will write V parallel Z square is equals to V perpendicular Zm square minus V perpendicular Z square. Now using this from the earlier expression this one. What we can write is V parallel Z square is equals to 2 Mu B naught by m times 1 plus Zm by a whole square minus 2 Mu B naught by m times 1 plus Z by a whole square. This is B. Now this is using B and this is using A that is all.

Doing some algebraic simplifications we will write V parallel Z square is equals to 2 Mu B naught by m times Zm by a whole square minus Z by a whole square. So, V parallel so the nonzero velocity component along the Z axis is this. This is the answer for the first question. Obtain the Z component of velocity. This is going to be the Z component of velocity.

So, the Z component of the velocity does depend on the position of the mirroring point with reference to the instantaneous position of the particle. Now let us look what is the second question average force acting on the particle. Now we have in the derivation itself we have realized that there will be a nonzero negative force which will act on the particle. So, we will solve it we will get the force as BZ is equals to B naught times 1 plus Z by a whole square. So, dou BZ by dou Z is equals to dou by dou Z of 1 plus whole square.

$$W = \sqrt{\frac{2\mu b_0}{ma^2}} \qquad W = 2\pi\nu$$

$$T = \frac{2\pi}{W}$$

$$\boxed{T = 2\pi a\sqrt{\frac{m}{2\mu B_0}}}$$

So, dou BZ by dou Z take the derivative inside you get 2 B naught by a square times Z. So, this is what is this is the dou BZ by dou Z. Now force from our discussion from the derivation we have force is minus mu times dou B by dou Z. So, using this let us say C using C the nonzero force is minus mu times 2 B naught by a square times C.

This is the answer for the second problem. Now we have to calculate the time period of oscillations of the particle which is executing a simple harmonic motion between the two mirroring points. Now we know the force which is M d square CX by dt square is Q times V cross B. So, this is F Z is minus 2 mu B naught by a square times Z. So, F is nothing but M d square Z by dt square is minus 2 mu B naught by a square times Z. So, d square Z by dt square is equal to minus 2 mu B naught by M a square Z.

So, this is just appearing in the form of d square Z by dt square plus omega square Z is equal to 0. What is this? This is a standard way we write a simple harmonic motion. So, these two are similar to each other. That means this term that appears as a coefficient to Z must be omega square.

So, omega square is 2 mu B naught by M a square. I will write the full relation plus 2 mu B naught by M a square Z is equal to 0. So, this has to be omega square. Omega is what is omega? Omega is the angle of frequency or I will write omega as 2 mu B naught by M a square. So, the time period that is associated with the simple harmonic motion is 2 pi by omega.

Omega is the angle of frequency, omega is 2 pi mu. So, this we know the value of omega. So, we can write the time period as 2 pi a square root of M by 2 mu B naught. So, the point is for the given configuration of the magnetic field, we are able to obtain the Z component of velocity and then we are able to obtain the average force acting on the particle along the Z axis, the average force which pulls the particle back and the period of oscillation, the simple harmonic motion. The particle will be trapped like this back and forth motion, the simple harmonic motion. So, this is one example where we can use the basic ideas with which we have understood the magnetic mirroring and solve some interesting problems.

So, we learned how the magnetic mirroring works and also learned how to solve numerical problems based on magnetic mirroring. So, the basic idea is the conservation of energy and the conservation of magnetic moment. Thank you.