Plasma Physics and Applications

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Week – 05

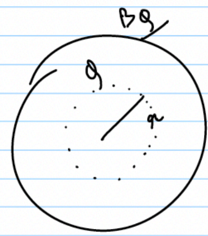
Lecture 24: Magnetic Mirroring - II

Hello dear students. So, we will continue our discussion on understanding the magnetic mirroring which helps us to confine plasma. So, this is the geometry that we have taken in which the magnetic field lines are converging into the positive z axis and by drawing the same radius at two different points we have realized the strength of the magnetic field at these two points will be same which means there is a gradient in this direction which is what we have written here the magnetic field gradient is parallel to the magnetic field itself. So, that means there is an invisible component of magnetic field along the R direction. So how do we get it? So let us say if we have to understand the magnetic field the fundamental law that magnetic field obeys is del dot B is equal to 0 the divergence of magnetic field is 0. So, cylindrical coordinates so we have a field like this something like this this is the R cap direction this is the z cap direction.

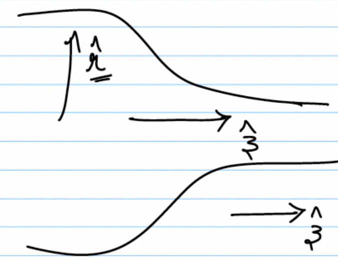
So the magnetic field now has an axial component and a transfer component. So this magnetic R component is not visible directly but you have to see carefully you will realize how the R component is existing. So, assuming del dot B is equal to 0 why I have used del dot B is equal to 0 because any configuration of magnetic field has to obey this. So now the geometry is we have B R we have B Phi and we have B z.

$\vec{\nabla} B \parallel \vec{B}$

$\vec{\nabla} \cdot \vec{B} = 0$



(B_r, B_ϕ, B_z)



$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

$+ve$

$\vec{\nabla} B \rightarrow$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

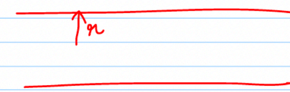
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\underbrace{\cancel{\frac{\partial B_\phi}{\partial \phi}}}_{0} + \frac{\partial B_z}{\partial z} = 0$$

$B = (B_r, B_\phi, B_z)$

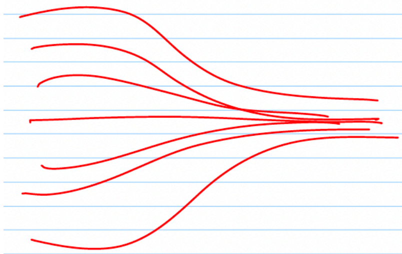
Of course the B Phi component is outright 0 we do not see any thing here B z is of course we have started with a non-zero B z component now B R is something which seems to be hidden in the plain sight but if B R is 0 or not has to be explained by this simple equation. So if I expand this divergence I will be able to write 1 by R dou by dou R of R B R plus dou B z by dou z is equal to 0. So I have not written the Phi component of the divergence equation because B Phi is 0 or change in B Phi along Phi is also 0 at a value of R. Now just look at this equation carefully if this equation has to be true there is no doubt that del dot B has to be 0 because by the fundamental nature of the magnetic field it has to be satisfied. So if it has to be 0 the way the magnetic field is changing with respect to z why is the magnetic field changing with respect to z because you have considered a configuration like this in which the magnetic field is converging into a smaller space as it travels towards the z axis that means there is a positive gradient in the magnetic field in this direction.

So if this dou B z by dou z is positive or if it exists it has to create this additional term. This additional term has to compensate this so as to make the sum as 0. So this is cause and effect kind of a thing. So since you have created a magnetic field like this it invariably produces a magnetic field component along R which again just goes ahead and tries to nullify dou B z by dou z just so that divergence becomes 0. So now this equation also gives us how we can get B R the invisible component B R in terms of the other things.

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0$$



$$r B_r = -\int_0^r r \frac{\partial B_z}{\partial z}\, dr$$



$$r B_r = -\frac{r^2}{2}\left[\frac{\partial B_z}{\partial z}\right] + c_1$$

Valid for all values of r

$r = 0 \Rightarrow c_1 = 0$ $\qquad \Rightarrow 0 = 0 + c_1 \Rightarrow c_1 = 0$

$$r B_r = -\frac{r^2}{2}\frac{\partial B_z}{\partial z}$$

$$\boxed{B_r = -\frac{\partial B_z}{\partial z}\cdot\frac{r}{2}} \quad \hat{r}$$

Let us say for instance I will write the entire divergence for the sake of clarity rather than just saying something will be 0 dou by dou R of R B R plus 1 by R dou B phi by dou phi plus dou B z by dou z is equals to 0. Now if you are wondering how did I write this equation from where did I write this equation. You have to refer to some basic vector algebra in which or you can just go to this electrodynamics book by Griffiths where you can see all the relevant formula of vector calculus are given in the first page itself where the divergence is written in all the coordinates. If you write the divergence in if you write the del operator in Cartesian coordinates it will be like this dou by dou x along i cap plus dou by dou y along j cap plus dou by dou z along k cap. So similarly the del operator can be written in cylindrical coordinates or in the spherical polar coordinates and those relevant formulae you can find anywhere.

$$m\frac{d^2 r}{dt^2} = q(\vec{V}\times\vec{B})$$

$$V = (V_r, V_\phi, V_z) = (V_r, V_\phi, V_z)$$

$$B = (B_r, B_\phi, B_z) = (B_r, 0, B_z)$$

$$B = (0, 0, B_z)$$

$$\nabla B$$

$$B = (B_r, 0, B_z)$$

$$\vec{F} = q(\vec{V}\times\vec{B})$$

$$\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ V_r & V_\phi & V_z \\ B_r & 0 & B_z \end{vmatrix}$$

Components of Lorentz force

$$F_r = q\left(\overset{1}{V_\phi B_z} - \cancel{V_z B_\phi}\right) \quad \text{— (1)}$$

$$F_\phi = q\left(-\overset{2}{V_r B_z} + \overset{3}{V_z B_r}\right) \quad \text{— (2)}$$

$$F_z = q\left(\cancel{V_r B_\phi} - \overset{4}{V_\phi B_r}\right) \quad \text{— (3)}$$

$$v_x^2 + v_y^2 = v_r^2$$

$$\left\{\begin{aligned} F_x &= q\,v_y B_z \\ F_y &= -q\,v_x B_z \\ F_z &= 0 \end{aligned}\right\}$$

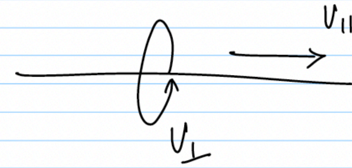
I have just used them here del dot B is equals to 0 and also using the fact that B is now defined as B R B phi and B z. Now dou B phi by dou phi you understand what is dou B phi by dou phi you take the cross section you fix the value of R let us say you fix any desired value of R if you draw this 360 degree arc from that point these different points will indicate different values of phi angles and we have created a magnetic field which is such that along this loci of points the magnetic fields B phi component will remain a constant. So this derivative outright becomes 0 what we are left with is 1 by R dou by dou R of R B R plus dou B z by dou z is equal to 0. What does it mean? If at all the magnetic field has to have a gradient along z axis then this component this term should be nonzero. If at all this dou B z by dou z has to be 0 it is impossible to have the divergence of gradient to be equal to 0 without this term being nonzero.

So we can write simply R B R is minus integral this derivative goes the other side becomes an integral is 0 to R R dou B z by dou z times dR. So simple vector simple calculus where I am just trying to separate B R or R B R is integral of R becomes R square by 2 which is minus R square by 2 times dou B z by dou z plus a constant of integration. So how do we get this constant? Let us say for all values of R including R is equal to 0. This expression is valid for all values of R. What is the meaning of this expression? Why is the radial component of the magnetic field so impossible to see in the plane sight? Because we have taken a configuration in which the magnetic field is just along the z axis.

$$F_z = -q V_\theta B_r$$

$$B_r = -\frac{1}{2} r \frac{\partial B_z}{\partial z}$$

$$F_z = -\frac{q V_\theta r}{2} \frac{\partial B_z}{\partial z}$$



$$V_\theta = \pm V_\perp$$

$$F_z = \mp \frac{1}{2} q v_\perp r_L \frac{\partial B_z}{\partial z}$$

$$= \mp \frac{1}{2} \frac{q v_\perp^2}{\omega_c} \frac{\partial B_z}{\partial z}$$

$$r_L = \boxed{\frac{m v_\perp}{qB}} \qquad \omega_c = \frac{qB}{m}$$

$$\boxed{F_z = \mp \frac{1}{2} \frac{m v_\perp^2}{B} \frac{\partial B_z}{\partial z}}$$

The radial component is something which comes into existence only because of the convergence of the magnetic field. Otherwise this component would not exist that is the meaning of this expression by the way. So, this expression is valid for so this is valid for all values of R. So, let us say we take R is equal to 0 is this valid when R is equal to 0 everything becomes 0 but the constant C1. So, C1 survives so how do you get this? This is very trivial we never discuss this sort of thing but let us say if you are still wondering so if I substitute R is equal to 0 I will get this 0 is equal to 0 plus C1 which means C1 is 0.

So, that means the constant we do not have to worry about the constant and we will go ahead and write RBR is minus R square by 2 dou Bz by dou z or BR is minus dou Bz by dou z into R by 2. Where did this minus come from? This minus came from the fact that the divergence has to be 0 that is why this term when taken to the other side gave us a minus. Now we have BR but most importantly BR seems to be just dependent on the magnetic field Bz and the way it changes with z and it also seems to be dependent on R you see larger the R larger will be the magnitude of B but there is a minus so these two things will not increase in the same direction rather they are correlated in the opposite directions. So, this is the beauty of the geometry that it spoke for itself in terms of this simple mathematics. So, what can we say? We can say that the magnetic field increases linearly with respect to R.

So, I can now safely write the direction for this as R cap. So, as you go along the geometry like this as you go along this with increasing values of R you are bound to get larger magnitudes of magnetic field. So, you go back to this cylindrical geometry you see here I will just draw it rather than just something like this. So, in this with increasing values of R you will get more and more stronger magnetic fields that is the meaning of this. Now having done all this now the magnetic field configuration has slightly changed.

So, ideally we expected this to be 0 0 B z because we thought the magnetic field only has the z component and coupled it with del B. Having added this del B into the configuration has now made it as B R 0 B z. You see that is the change that it has got along. So, this is B R nonzero just because del B exists. Now let us try to go back to our equation of motion.

What is the equation of motion? m d square x by dt square is let us say I will not use x rather I will use R is Q times V cross P. So, V is V R V phi V z and B is V R V phi V z. So, you can write V R V phi V z the process the particle has no limitation detail it can have a velocity along any direction. But the magnetic field has some limitations where we have restricted the magnetic field to have an indirect radial component a 0 phi component and a direct z component. So, we can write the force as Q times V cross P or we can write Fx using this using this we can write Fx as Q times V phi V z minus V z B R Fy as Q times minus V R B z plus V z B R F z Q times V R B phi minus V phi B R.

So, let us say we call this equation as 1, 2 and 3. So, B phi is 0. So, this term becomes 0 and this term also becomes 0. Now how many terms we have? We have 1, 2, 3, 4. What are these? These are the components of Lorentz force.

Now let us put this in contrast to the magnetic field only along z axis without any convergence or without any gradient where we have Fx as Q times Vy Bz and the Fy is minus Q times Vx Bz and Fz is simply 0. So, this actually we have obtained at the beginning of the last class. So, if you are wondering what are these equations? These equations are valid for a particle which is in a magnetic field along one direction. So, this magnetic field is not changing its magnitude along a particular direction but whereas the configuration shown in equations 1, 2 and 3 has a magnetic field along z axis also has a gradient along the z axis that is why they are different. But the contrasting difference between these two equations is that what is the type of motion that the particle executes under this force? Can you make a guess? Fx and Fy if you put Fx and Fy you will realize that it will give you something like Vx square plus Vy square is something like in a measure of RL square.

It should be x square plus y square is equals to RL square if the center is at the origin. But the point is Fx and Fy the force components Fx and Fy will just make the particle gyrate along the axis along the magnetic field. It will gyrate along the magnetic field. So, it is not going anywhere you see it is just gyrating. If you put an electric field the particle will move along the direction of electric field.

But Fz on the other side Fz is simply 0. So, the magnetic field is not able to move the particle along the z axis Fz is 0. But what is it doing it is just executing the circular motion which is defined in the xy plane. You see Fx and Fy are nonzero and they seem to be dependent on the components of velocity which are Vy and Vx. But you compare this to equation 1, 2 and 3.

Fx and Fy will still be continuing to give circular motion no doubt. But Fz seems to be in the opposite direction you see. Let us write Fz. So, Fz is equals to minus q P phi Br we know Br as minus half r dou Bz by dou z. So, Fz is minus q V phi r by 2 dou Bz by dou z.

So, this 1 minus comes because of the charge q. Now if this is your z direction let us say if the particle is directing like this. This velocity component becomes V parallel and this velocity component is V perpendicular. So, V perpendicular has the x and y components Vx square plus Vy square whereas V parallel is simply Vz square. So, we can safely assume that the r and phi components will still continue to give circular motion.

So, Fz V phi is plus minus V perpendicular depending on the direction of variation. So, Fz can be written as minus plus because there is a minus plus minus becomes minus plus after multiplying with minus becomes half q V perpendicular r L dou Bz by dou z. Substituting something that we know already half q V perpendicular by omega dou Pz by dou z. Again r L is equals to m V perpendicular by q B or m by q B is 1 by omega C or omega C is equals to q B by m. So, we can write Fz is minus plus half m V perpendicular square by B dou Bz by dou z.

So, this force Fz seems to be dependent on this m V perpendicular by 2B. We have discovered that this is nothing but the magnetic moment in the beginning of the last class. So, Fz becomes minus mu dou Bz by dou z. So, at this point what have we learnt? We just have taken some configuration of magnetic field and in that configuration we realized the particle will experience an additional force along the negative z axis. So, this nonzero force along the z axis is responsible for trapping of the plasma particle in a given magnetic field or in a given configuration of magnetic field which we call as the bottle configuration something like this.

We will stop here and we will continue the remaining part of the magnetic mirroring in the next class. Thank you.