Plasma Physics and Applications

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Week – 05

Lecture 22: Numerical Problems on Drifts

Hello dear students. In today's class we will try to solve some numerical problems based on plasma drifts single particle theory. So, we will try to solve this numerical problems and in the process we will also try to learn the underlying physics which is important for solving this problems. So, this is the problem number 1. So, let us see what this problem statement is and try to understand how we can solve it. So, the given question is suppose the Earth's magnetic field is about 3 10 to the power of minus 5 Tesla at the equator and falls off as 1 by R cube as for a perfect dipole.

So, that means the magnetic field of the Earth at the equator is given and with the distance as you move away from the surface or from the center it falls off as 1 by R cube where R is the distance from the center like a perfect dipole. You can think of a dipole magnetic field you will if you know that expression for magnetic field away from a dipole in terms of R and in terms of theta you will remember that it is 1 by R cube dependence. Let there be an isotropic population of 1 eV protons and 30 keV electrons. So, that is the protons have this energy 1 eV they must much less energetic and 30 keV electrons.

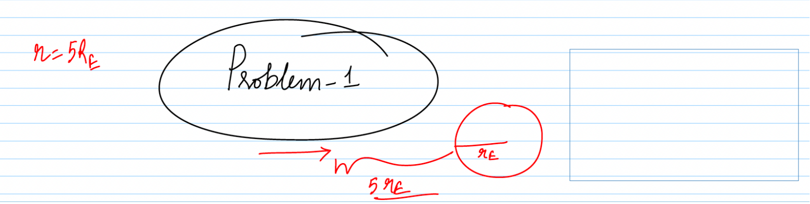
Suppose the earth's magnetic field is 3×10^{-5}T at the equator and falls off as $1/r^3$ as for a perfect dipole. Let there be an isotropic population of 1-eV protons and 30 keV electrons, each with density $n=10^7$ m^{-3} at r=5*earth radii in the equatorial plane.

(a) Compute the ion and electron $\vec{\nabla}$ B drift velocities.

(b) Does an electron drift eastwards or westwards ?

(c) How long does it take an electron & ion to encircle the earth ?

(d) Compute the ring current in A/m^2



Electrons being lighter also seem to have higher energy that is 30 keV each with densities 10 to the power of 7 per meter cube at R is equals to 5 times the radius of the Earth. So, we will write R is 5 Re. So, you take the Earth the plasma is coming towards the Earth and we have to perform our calculation at a distance of if this is Re this distance is 5 times Re in the equatorial plane. Now what we have to do is we have to compute the ion electron del B drift gradient drift section del B drift velocities and we have to see how the electron will drift in comparison to the ions when we when they encounter this gradient and how long does it take for an electron let us say for an electron and ion to encircle the Earth and compute the ring current in ampere per meter square. Now this topic is already being taught we know the basics of this problem that means we have realized when the plasma sees or experiences a gradient magnetic field that means a magnetic field whose strength is changing with respect to space.

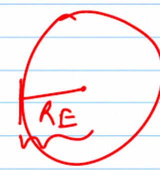
Then the plasma particles or the electrons and ions will experience a drift velocity which is called as a gradient drift velocity and this gradient drift velocity seems to be a function of the charge that means the direction of the drift velocity will be different for electrons and ions that is the negative charges and the positive charges. So, we have derived expression for the gradient drift velocity we have to use that expression or those formula and we have to get this problem done. So, let us start with what we are given how we can do it. So, the magnetic field B is proportional to 1 by R cube. So, we can write the magnetic field B as a constant by R cube.

So, the constant magnetic field at the beginning is 0.3 10 to the power of minus 4 what is this 10 to the power of minus 4 Tesla it is given to 3 10 to the power of minus 5 Tesla. So, that is what I have written and this is at a distance of Re from the surface from the center. So, this magnetic field is at the surface and one earth radii how much is Re? Re is 6400 kilometres. So, for a distance of Re if this is 0.

$$B \propto \frac{1}{r^3} \Rightarrow B = \frac{c}{r^3} = 0.3\times10^{-4}\ \text{Tesla}$$

$$B = \frac{0.3\times10^{-4}}{(r/R_E)^3} = \frac{0.3\times10^{-4}}{(5R_E/R_E)^3} = 2.4\times10^{-7}\ \text{Tesla}$$

$$\vec{B}(5R_E) = 2.4\times10^{-7}\ \text{Tesla.}$$



$R_E = 6400\times10^3$

$5R_E$ $\propto \frac{1}{r^3}$

$$v_{\nabla B} = \frac{1}{2} v_\perp r_L \left| \frac{\vec{B}\times\nabla\vec{B}}{B^2} \right|$$

$$r_L = \frac{m v_\perp}{qB}, \quad \omega = \frac{qB}{m}$$

3 10 to the power of minus 4 Tesla we have to find out what would be the magnetic field for a distance of 5 Re. Assuming that the magnetic field will not decay linearly it will decay with a factor of 1 by R cube. So, now we will use the distance the B a factor of let us say 0.3 10 to the power of minus 4 divided by the distance R is to be measured in the distance of in the units of Re. R is the distance which is given to be in the units of the radius of the earth.

$$v_{\nabla B} = \frac{1}{2} v_{\perp} r_L \left| \frac{\vec{B} \times \vec{\nabla} B}{B^2} \right|$$

$$B = \frac{c}{r^3}$$

$$v_{\nabla B} = \frac{1}{2} v_{\perp} r_L \left| \frac{\vec{\nabla} B}{B} \right|$$

$$\vec{\nabla} B = \frac{\partial B}{\partial r} \hat{r} = -3 \frac{c}{r^4} \hat{r} = \frac{3}{r} \left(\frac{c}{r^3} \right) (-\hat{r})$$

$$= \frac{3}{r} \vec{B} (-\hat{r})$$

$$\vec{\nabla} B = \frac{3}{r} \vec{B} (-\hat{r})$$

$$\boxed{\left| \frac{\vec{\nabla} B}{\vec{B}} \right| = \frac{3}{r}} \quad 5R_E$$

So, that is why I have put R by Re raise to the power of 3. So, we can write it as 0.3 10 to the power of minus 4 Tesla divided by how much is R it is 5 Re by Re whole cube which will be equal to 2.4 10 to the power of minus 7 Tesla. What is this? This is the amount of magnetic field that is going to be available at this distance how far is it? It is 5 Re.

So, it is like if something falls off to this value at this distance then you are asked to find out what will be the value at this distance it is a linear proportion generally, but what we are given is it does not fall off linearly rather we cannot use the simple proportionality here because we are given that the decay is not linear rather it is 1 by R cube. So, what is this? This is magnetic field available at 5 Re. Why is 5 Re specific? Because 5 Re is where the plasma is situated right now and if you go back to the question once again we have to find out what will be the drift velocity that will be experienced by the electrons and ions when they start experiencing the magnetic field at 5 Re. So, the magnetic field magnitude B at 5 Re is 2.4 10 to the power of minus 7 Tesla.

$$v_{\nabla B} = \underbrace{\frac{1}{2} v_\perp r_L}\left(\frac{3}{r}\right) \qquad r_L = \frac{m v_\perp}{qB} \; ; \; \omega_c = \frac{qB}{m}$$

$$\frac{1}{2} v_\perp r_L = \frac{1}{2} v_\perp \frac{m v_\perp}{qB} = \frac{1}{2} \frac{m v_\perp^2}{qB} = \frac{1}{2} \frac{v_\perp^2}{\omega_c}$$

$$\frac{1}{2} v_\perp r_L = \frac{1}{2} \frac{v_\perp^2}{\omega_c} \qquad \text{— (a)}$$

$$\frac{1}{2} m v_\perp^2 = k_B T \implies v_\perp^2 = \frac{2 k_B T}{m}$$

$$\rightarrow \frac{1}{2} \frac{m v_\perp^2}{qB} = \frac{1}{2} \frac{v_\perp^2}{\omega_c} = \frac{1}{2} \frac{2 k_B T / m}{qB/m}$$

$$\frac{1}{2} \frac{v_\perp^2}{\omega_c} = \frac{(k_B T)_J}{qB} = \frac{(k_B T)_{eV}}{B}$$

Now let us bring the formula for the gradient drift which is half V perpendicular RL times V cross del B by B square. What is this? This is V del B. Where is charge? Charge is there in this. What we can write RL as m V perpendicular by q B omega as q B by m. So, in RL it is there.

The charge is there. Now, let us use this formula. So, we have to find out the drift velocity V del B. How is it given? It is given half V perpendicular RL times B cross del B by B square. You can simplify it as half V perpendicular RL del B by B the magnitude the absolute value of del B by B.

Let us find out what is del B by B. So, we have how can we write del B? del B as dou B by dou R along R cap. Using B using from the earlier formula using the fact that B is equals to C by R cube we can write this as dou B by dou R. A derivative of this with respect to the radius will give you minus 3 into C by R power 4 along R cap. So, this is 3 by R let us say into C by R cube.

Since there is a minus let us say we write it as minus R cap. So, 3 by R remains a factor that is outside and C by R cube from here is B itself is B along minus R cap. So, the gradient of B del B is 3 by R times the magnetic field which is acting along minus R cap. What have we done? We have just evaluated del B. So, the if we now rearrange this in terms of something that we actually need for evaluating.

So, del B by B will be just to bring B into the denominator will just be 3 by R. We have gotten rid of the minus sign by using the absolute value the magnitude just the magnitude. So, it may be confusing just to follow this I will go back what I have done so far is I know the magnetic field is proportional to 1 by R cube that means it is falling off with 1 by R cube as we go away from R. So, there is a gradient I mean that is I mean the magnetic field is changing as we go along the distance R equatorially. Now we are given what is the magnetic field at the surface in the equatorial plane and with an R cube dependence we have to find how it scales at 5 Re that we have done here.

It turns out that the magnetic field that is going to be available equatorial at the equatorial plane at 5 Re is 2.4 10 to the power of minus 7 Tesla. Now once this is done we pulled the gradient drift velocity formula from our earlier classes. So, V del B is half V perpendicular R L B cross del B by V square just to be familiar with this formula we have written them again here. Now we can reduce it into this formula V del B is equals to half V perpendicular R L del B by B the magnitude of del V by B.

So, we have evaluated del B by B del B is dou B by dou R along R cap since we know the form for B we have used this differentiated it once with respect to R we got this we slightly rearranged it. So, that we have the B appear again in the right hand side brought it to the left hand side and now we have del B by B as 3 by R. So, magnetic field the change in the magnetic field or del gradient which tells you the direction the gradient tells you the increasing direction of the physical parameter. So, if the magnetic field is increasing as you go towards the earth. So, you have to take the radius R cap in the minus.

$$\vec{v}_{\nabla B} = \underbrace{\frac{1}{2} v_\perp r_L}\left(\frac{3}{r}\right)$$

$$= \frac{(k_B T)_{eV}}{B} \times \frac{3}{r}$$

$$\vec{v}_{\nabla B} = \underbrace{\frac{(k_B T)_{eV}}{2.4\times10^{-7}} \times \frac{3}{5\times6400\times10^{3}}}_{} \to 0.39$$

$$\boxed{\vec{v}_{\nabla B} = (0.39) * (k_B T)_{eV}}$$

So, this minus is just telling you as you go towards the earth you have increasing magnetic field strengths. So, del B by B is 3 by R. Now we have everything. So, we have to calculate V perpendicular R L then if you now R is equals to R e times 5 because you have to perform the calculations at 5 R e. Now we just have to substitute everything into this formula.

So, what is the formula that we will actually need is V the gradient drift velocity V del V B V del B is half V perpendicular R L into 3 by R. Now let us talk about this parameter half V perpendicular R L. So, R L is mV perpendicular by q B half V perpendicular mV perpendicular by q B which is half mV perpendicular square by. So, that means we can write half V perpendicular square and omega C is q B by m. So, this m by q B can be written as 1 by omega C.

So, this is half V perpendicular R L half V perpendicular R L is half V perpendicular square by omega C. Now see from the introductory aspects of the plasma physics we know that plasma temperatures are generally written in electron volts. So, half mV square is k B T that means V perpendicular square the kinetic energy equal to the thermal energy V perpendicular is equals to 2 k B T by m. Now again using this relation let us say we call this relation as a using this relation we can write half mV perpendicular square by q B or half V perpendicular square by omega C is equals to half V perpendicular to 2 k B T by m. So, this is 2 k B T by m divided by omega is q B by m.

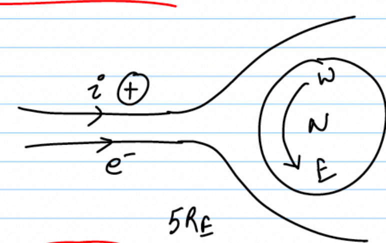
So, all of this this is half mV perpendicular square by omega C is nothing but k B T by q B. What have we done if you are just wondering what I have done this is from up to here it must be clear to you. So, now I am looking at this factor half V perpendicular R L using the definition R L I am just writing half V perpendicular R L as half V perpendicular square by omega C. What is V perpendicular? The perpendicular component of the velocity. Now this is half V perpendicular by omega C the kinetic energy mV perpendicular half mV perpendicular square is equals to k B T.

(1) (a) $V_{\nabla B_p} = 0.39 \times (k_B T)_{eV} = 0.39 \times 1\ eV = 0.39\ m/s$

$V_{\nabla B_{e^-}} = 0.39 \times 30 \times 10^3 = 11{,}700\ m/s$

(b) Ions = Westward

e^- = Eastward



(c) Time =

$$2\pi r = 2 \times 3.14 \times 5 \times 6400 \times 10^3 = 2 \times 10^8\ m$$

$$t_p = \frac{2 \times 10^8}{0.39} = 5.12 \times 10^8\ s = 142450\ \text{hours}$$

So, we can write V perpendicular square as 2 k B T by m. I have used V perpendicular square into this formula. So, now I have got this. So, this entire factor which is multiplying 3 by R is equal to k B T by q B. Now k B T is of course the energy which is obviously in joules we have not made any conversion.

So, k B T is right now in joules. Now if I divide this energy by the charge of electron which is 1.6 10 to the power of minus 19 coulombs we will get we can write k B T not in joules rather we can write k B T the energy in electron volts divided by B. This q will no longer exist in the denominator because it is dividing the energy and converting the energy units from joule to electron volts. Let us again try to write everything. So, we have the gradient drift velocity V del B as half V perpendicular RL into 3 by R.

So, all of this is now simply k B T in electron volt divided by the magnetic field into 3 by R. So, let us substitute the value of R V del B k B T into eV. The magnetic field is 2.4 10 to the power of minus 7 into 3 divided by 5 into 6400 kilometers 6400 into 10 to the power of 3 meters. Now k B T is the energy of plasma particle in electron volts.

If you go back to the question you will see that the protons have an energy of 1 electron volts the positive charges. And the electron the negative charges have an energy of 30 keV. So, it is already existing in the units of electron volts. So, we can use the energy directly into this formula which goes here. So, the remaining factors can be simplified as V del B equal to 0.

39 times k B T in electron volts. So, this is the reduced form of the gradient drift velocity formula. It is the same. How did I get 0.39? 0.39 is just algebraic simplification of this entire factor 3 by 2.

$$t_{e^-} = \frac{2\times10^{8}}{11700} = 4.748 \text{ hours}$$

(d) $$\vec{J} = ne\,\vec{v}_{\nabla B_{e^-}}$$

$$= 10^{7}\times1.6\times10^{-19}\times1.17\times10^{4}$$

$$\vec{J} = 1.872\times10^{-8}\ \text{A/m}^2$$

4 10 to the power of minus 7 divided by yeah this entire factor will be equal to 0.39. Now once we have done this let us now go back to the calculation one after the other. So, let us say the first thing that we are asked is to calculate the electron and ion drift velocities. So, question number 1 is or part V del B of protons is 0.

39 into k B T in eV. So, which is 0.39 into 1 eV which is 0.39 meters per second. So, I should have written here V del B. This is in meters per second and V del B for an electron which is 0.

39 into the energy is 30 k eV. So, 30 10 to the power of 3 which is 11,700 meters per second. So, this is the difference in the gradient drift velocity between electron and proton. So, the difference is always because of the higher energy that is possessed by the

electron and also the smaller mass that it has. Now if you look at this picture now the plasma is coming from here like this. Let us say this is ion or the positive charge and this is electron.

So, because of the charge dependence of the gradient drift velocity both the particles, so this is the north you are looking from the top. So, the positive charge particles, so the magnetic field can also be seen in this way. The magnetic field of the earth if you consider that and use the right hand thumb rule we can realize that the ions will move like this and electrons will move like this. So, ions will go westward. So, in this picture it is very important that the earth is evolving like this.

This is the west and this is the east. Ions will move westward and the electrons will move eastward. So, this differential movement of the charges is responsible for the creation of ring current. We know that. Now since the particles have separated they are encircling the earth. How long does it take for the particles to complete one revolution of the earth? There is the question.

Go back see the stem of the question. It says how long does it take for electron and ion to encircle the earth? Let us say we consider just at the at one earth radii or at five earth radii 5 Re is when they start experiencing the drift. So, time we have to calculate the time. So, s is equals to ut distance is equals to velocity times the time. So, the distance is 2 pi r which is 2 into pi into r.

What is r? 5 times 6410 to the power of 3. This will be something like 2 10 to the power of 8 meters. So, the time that is taken for the proton because we know the velocity we can simply calculate distance by velocity will give you the time. So, this is 2 10 to the power of 8 divided by velocities 0.

39 very small velocity. This will take 5.12 10 to the power of 8 seconds which will be approximately equal to 1423. 450 hours. So, for the proton for the solar proton which is coming with energy of 1 electron volt, it will take nearly 142450 hours to complete one revolution at 5 times the radius of the earth. And if you consider the electrons similarly the time taken for the electrons to complete revolution it will be 2 10 to the power of 8 divided by 11700 which will be equal to 4.

74748 hours. The electron will move very fast that is why they will constitute they will finish the circumference within a very short period of time. Now, the last part of this question is question b calculate what is the ring current density. So, obviously, you these particles are rotating around the earth. So, the electrons are moving in a in a circular path there they will generate a current the it will be responsible to create a current density J is equals to N e V del B electron. I should also include the ion, but you have seen the time it takes for the ion to complete one rotation is very very very very large that means we can neglect the current density contribution from the ion.

So, we can simply write J is equals to N e V del B. So, it is N is 10 to the power of 7 particles per unit volume times the charge which is 1.6 10 to the power of minus 19 coulombs for the electron and the velocity that we have calculated is 1.17 10 to the power of 4. So, the current density the ring current density will be 1.

872 10 to the power of minus 8 ampere per meter square. So, this is how we solve a problem to calculate the gradient drift velocity and the associated physical parameters which come into existence because of the gradient drift. So, first we have calculated first we have calculated the velocities the drift velocities the direction of moment of the charges and then the amount of time that is required for the positive charge proton and the electron to cover one rotation of the earth at five times the radius of the earth and then we have calculated the resulting ring current density. So, plasma physics is a highly mathematical course and the objective of this course is also to develop problem solving ability in all the students. I think this problem will be a very fine example of how we can arrange all the various physical variables that we need to solve a numerical problem. Thank you.