Plasma Physics and Applications

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Week – 05

Lecture 21: Vacuum Drift

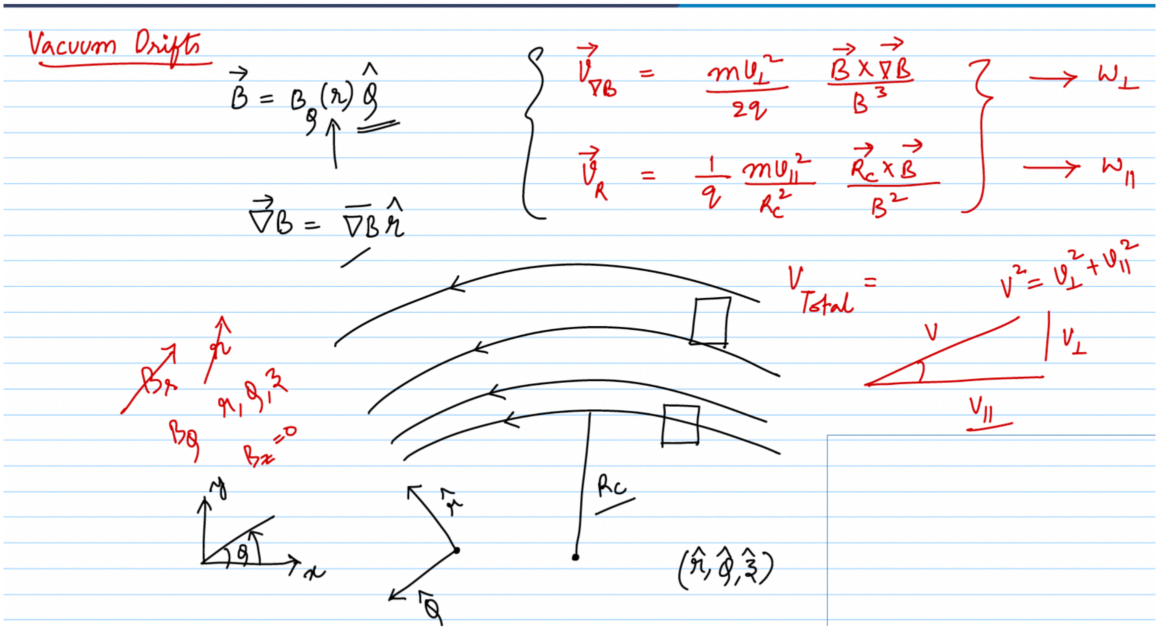
Hello dear students. In today's class we will continue our discussion of obtaining a combined expression for gradient and curvature drifts. So, in the last class we have seen how an expression for gradient drift and curvature drift can be obtained. So, V del B which is the velocity due to gradient of the magnetic field is m V perpendicular square by 2 q times B cross del B by B q. And the curvature which is called as a V r the curvature drift velocity is V r is equals to 1 by q m V parallel square by r c square r c is the radius of curvature times r c cross B by B square. So, these two expressions are valid for a realistic magnetic field which means any magnetic field will have a gradient and also a curvature.

Now we are going to combine these two expressions and write the total drift that the particle will experience whenever it enters into a magnetic field. So, if you look at these two the first term seems to be affecting the perpendicular kinetic energy the second term seems to be affecting the parallel kinetic energy. So, we have seen if you consider a particle to have a velocity V so the perpendicular component is this and this is the parallel component. So, it is V square is V perpendicular square plus V parallel square.

Now this discussion is going to be referred to as vacuum drifts. Now in order to combine these two terms the first difficulty is that let us say even if we if we ignore whatever appears outside the curl in each term we have one curl between B and the grad B and the other term between r c and B. So, we have to find out a way wherein both of these curls can be written in a similar way. So, to do that we will consider the cylindrical geometry of a magnetic field and we will try to see how these two terms can be written in the same way. So, let us say we consider the magnetic field like this which is curved in nature like this and with respect to the center r c is the radius of curvature of the magnetic field.

The magnetic field lines are going or curving towards let us say towards the left. You consider a coordinate system such that this direction is phi cap the direction in which the magnetic field lines are going towards is phi cap. This is the radially outward this one is the radially outward direction this let us say we call it as r cap and this is the z cap. So, we have r phi z coordinate system we have r cap phi cap and z cap. If you want to translate it to a Cartesian coordinate system you have to assume that you define the angle phi such that it is increasing when you measure it away from x axis and towards the y axis the positive y axis.

So, you are taking the magnetic field lines along phi. So, they are rotating along parallel to phi. So, this is the equivalent when you draw it to a Cartesian coordinate system. So, we can write the magnetic field B to be equivalent to B phi as a function of r along phi cap. So, what does it mean? magnetic field is along phi cap it is changing along r.



So, it has a gradient along r. So, as you go along r c it is changing its magnitude is changing. So, for instance you take a unit area here or a unit area here the number of flux lines that will cross that will go through this unit area will be different there in indicating a gradient to be present in this direction. So, we write the del B is equals to del B is the magnitude for example is along r cap. So, the magnetic field is directed along phi cap and it is changing along r cap I hope it is clear.

So, if you take such a magnetic field and then we have all the variables we have established what is del B what is r c. So, we are now trying to see how these two terms can be combined into a single expression. So, let us take some hint from the cylindrical coordinate system if you have del cross B you can write del cross B in the cylindrical coordinate system as 1 by r dou B z by dou phi minus dou B phi by dou z all of it along r cap plus dou B r by dou z minus dou B z by dou r along phi cap plus 1 by r dou by dou r of r B phi minus dou by dou phi of B r along z cap. So, now the magnetic field is B r B phi B z this is the cylindrical components of the magnetic field vector. But what we have made a specific choice of magnetic field which is like it has gradient along the radial component and it is directed towards the phi cap.

If you substitute the magnetic field configuration that we have taken we can write. So, if you are wondering how I got this big expression this is from the you can refer to any standard book which gives the curl of a vector quantity in cylindrical coordinates. So, this is the curl of a vector quantity in cylindrical coordinates. This is pretty standard. So, I have just used the form as it is and I have substituted the components of B that I have taken for this picture.

curl of a vector quantity in cylindrical coordinates

$$\bar{\nabla}\times\bar{B} = \left(\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial}{\partial \phi}B_r\right)\hat{z}$$

$$\vec{B} = (B_r, B_\phi, B_z)$$

$$\bar{\nabla}\times\bar{B} = \left(\frac{1}{r}(0) - 0\right)\hat{r} + (0-0)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\phi) - 0\right)\hat{z}$$

$$\bar{\nabla}\times\bar{B} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\phi\right)\right]\hat{z} \leftarrow$$

Maxwell equations

$$\bar{\nabla}\times\bar{B} = \mu_0\left(J + \epsilon_0\frac{\partial E}{\partial t}\right)$$

$$\text{Vacuum} \Rightarrow \vec{\nabla}\times\vec{B} = 0$$

So, if I substitute B r B phi and B z values according to the configuration that I have described here what I can write is it is going to be 1 by r times 0. What does it mean? B z has no variation along phi along the phi direction it is not varying that is why the derivative is 0 minus 0 and B phi is also not varying along z direction plus. Just try to match each of these and so that we can understand what is going on. You see this you go back one thing is very clear magnetic field component B r has a variation B r changes when r changes it can be it need not be directly proportional, but B r has a variation, but B r has a variation B r changes when r changes and B phi is not changing it is a constant thing. So, B phi is not changing when r is changing when phi is changing when z is changing and B z is itself is 0 actually.

So, if you bring this into the picture so you see dou B z by dou phi is 0 because that means the z component of the magnetic field itself is 0. So, there is no point of taking a derivative with respect to phi B phi dou B phi with respect to z with respect to z of course it is not changing that is why it is 0 B r by B z dou B r by dou z. So, the radial component of the magnetic field of course does not have any variation with respect to the z coordinate. So, just for ready reference I am drawing the coordinate system here. So, this direction is phi cap out of the board is z cap and this is r cap and dou B z by dou r B z itself is 0.

$$\bar{\nabla}\times\bar{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = 0$$

$$\Rightarrow rB_\phi = \text{Constant} = k$$

$$\Rightarrow B_\phi = \frac{k}{r}$$

$$\frac{\partial B_\phi}{\partial r} = -\frac{k}{r^2} = -\frac{B_\phi}{r} \Longrightarrow \boxed{B_\phi = \frac{k}{r}\hat{\phi}}$$

at $r = R_c$

$$\boxed{B_\phi = \frac{k}{R_c}\hat{\phi}}$$

Ratio

$$\frac{\nabla|B|}{|B|} = \frac{-\frac{k}{R_c^2}\hat{R}_c}{\frac{k}{R_c}}$$

So, the derivative will obviously be 0. B phi is of course r B phi is changing with respect to r why because so if you take B phi is a coordinate which is going here B phi is this component. So, if you take different values of r and measure B phi you will get different B phi different magnitudes. So, that means that B phi is changing with respect to r. So, that is why I have retained this derivative as it is and the radial component of the magnetic field is not changing with respect to phi.

So, at any value of phi B r will be constant. So, B r this will be 0. So, this is the breakdown of all the terms inside this curve. So, simplifying we will write it simply as del cross B is equals to 0 0 1 by r dou by dou r of r B phi and this entire thing is along z cap. Now from the Maxwell equation we can write del cross B is mu naught J plus epsilon naught dou E by dou t.

$$\frac{\nabla |B|}{|B|} = \frac{-\frac{k}{R_c}\hat{R}_c}{\frac{k}{R_c}} = \frac{-\hat{R}_c}{R_c} = -\frac{\vec{R}_c}{R_c^2}$$

$$\Longrightarrow \boxed{\nabla |B| = |B| \left(\frac{-\vec{R}_c}{R_c^2}\right)} \quad \text{——} \; (1) \qquad \hat{R}_c = \frac{\vec{R}_c}{|R_c|}$$

$$v_{\nabla B} = \pm \frac{1}{2} \frac{v_\perp r_L}{B^2} (\vec{B} \times \vec{\nabla} B)$$

$$v_{\nabla B} = \mp \frac{1}{2} \frac{v_\perp r_L}{\cancel{B^2}} \vec{B} \times \cancel{|B|} \frac{\vec{R}_c}{R_c^2} \qquad [\text{Using } (1)]$$

$$v_{\nabla B} = \mp \frac{1}{2} \frac{v_\perp}{B} \left(\frac{m v_\perp}{qB}\right) \frac{\vec{B} \times \vec{R}_c}{R_c^2}.$$

$$v_{\nabla B} = \mp \frac{1}{2} \frac{v_\perp^2}{\omega_c B} \frac{\vec{B} \times \vec{R}_c}{R_c^2}$$

So, this is what we know from the Maxwell equation, but for the configuration of the magnetic field that we have considered we have realized that del cross B is equal to this. Now let us say we consider vacuum where there is no free current density, there is no electric field. So, in that case we can write del cross B the curl of magnetic field will be 0 in vacuum because we will say that there are no current densities which can create a magnetic field. So, in that case what we can do is Maxwell equations in vacuum expect or require del cross B is equals to 1 by r dou by dou r of r B phi is equals to 0 or we can infer that r B phi is going to be a constant as long as you are trying to look at the change along r. So, we can write which implies B phi let us say we assume this constant to be k B phi is equals to k by r.

Now if B phi is k by r dou B phi by dou r is equals to minus k by r square that is equals to minus B phi by r. So, just look at this B phi the rate of change of B phi with respect to r is equals to minus k by r square. So, as you increase the value of r what is happening to the B phi we can refer to that B phi is decreasing. So, this is a minus. That means that you have the gradient how do you have the gradient? Gradient is in this direction as you are moving in this direction with increasing r B phi is going to be smaller.

$$\vec{v}_{\nabla B} = \mp \frac{1}{2}\frac{v_\perp^2}{\omega_c B}\,\frac{\vec{B}\times\vec{R_c}}{R_c^2}$$

$$= \frac{1}{2}\,\frac{m v_\perp^2}{q B^2}\,\frac{\vec{B}\times\vec{R_c}}{R_c^2}$$

$$\vec{v}_{\nabla B} = \frac{W_\perp}{q B^2}\cdot\frac{\vec{B}\times\vec{R_c}}{R_c^2} \quad \text{—— (2)}$$

$$\vec{V}_R = \frac{m v_\parallel^2}{q B^2}\cdot\frac{\vec{R_c}\times\vec{B}}{R_c^2} \quad \text{—— (3)}$$

(2) & (3) can be combined

$$\boxed{V_{total} = \frac{m}{q}\left[v_\parallel^2 + \frac{v_\perp^2}{2}\right]\frac{\vec{R_c}\times\vec{B}}{R_c^2 B^2}}$$

So, as you move larger distances you will encounter smaller values of B phi which means that your magnetic field was strong in the beginning or you can simply infer that the gradient is in this direction del B is in this direction. So, this is one inference that comes directly out of this. We now have a form for the magnetic field along this particular direction. So, B phi is equals to which implies we can write B phi is equals to k by r phi cap. So, all the description that we have taken in the beginning is very well agreeing with the form of magnetic field that we have obtained.

Something that is a clear validation is it is along phi cap and it is k by r. So, with increasing distance of r the magnitude of B phi is changing or decreasing. Let us say at r is equals to r c we can write B phi is equals to k by r c along phi cap. Let us say we at r is equals to r c this is the value. Now let us take a ratio we are just considering a ratio which we will probably use in some other form.

Let us say we have a ratio of del B by mod B which is equals to using this B phi relation we can write it is minus k by r c square along r c cap by k by r c. Let us say we take this further what is it del B by mod B is equals to at r is equals to r c. It is quite obvious you see del B being directed along r cap but not phi cap that is the basic nature B phi is along phi cap but del B has to be along r cap or at r c it is along r c cap. So, upon algebraic simplification we will get it as minus r c cap divided by r c or if you use the definition of a unit vector r c cap is r c vector divided by mod r c we can write it as minus r c by r c square. So, from this ratio we can write grad B as B times minus r c by r c square.

So, this is what we are going to use further. So, we have V del B from the earlier expression plus minus half V perpendicular r L by B square times B cross del B. How you may ask? This is the relation that we have already derived B cross del B. Now, using del B for this formula we can write V from this ratio let us say we call this as equation number 1 equals to minus plus half V perpendicular r L by B square times B cross mod B r c by r c square. What have I done? I have using equation 1.

So, del V del B is now going to be minus plus half V perpendicular r L is m V perpendicular by q B B cross r c by r c square. And you have one more B V perpendicular by B or V del B is equals to minus plus half V perpendicular square divided by omega c times B cross r c by r c square. So, this is using 1. So, we have V del B is equals to minus plus half V perpendicular square divided by omega c B B cross r c by r c square. So, this m by q B is what is written as omega c.

For if you are confused how did I get omega c is how I get it. Now going back so let us say we use this relation V del B is equals to minus plus V perpendicular square by omega c B times B cross r c by r c square. Rearranging the terms again we will get half m V perpendicular square by q B square times B cross r c by r c square. So, this is W perpendicular the perpendicular kinetic energy by q B square into B cross r c by r c square. Now if you remember the earlier expression for the curvature drift.

So, now we had this B cross del B what we have done by considering this geometry and making appropriate substitutions we have written B cross del B in terms of as B cross r c. So, that it matches with our expression for the curvature drifts. Now we will bring the curvature drift V r is equal to m V parallel square by q B square times r c cross B by r c square. So, we can now let us say we call this equation as 2 and 3. Now we can combine 2 and 3 can be combined and we write V total this is vacuum drifts this is going to be written as m by q times V parallel square plus V perpendicular square by 2 times r c cross B divided by r c square B square.

So, this is the total drift that the particle experiences when it sees a realistic magnetic field. This expression is also referred to as the total drift or vacuum drift why because we have taken del cross B as 0 which is valid for vacuum when there are no additional current densities and electric fields. Now if you consider this to be the total velocity in the particle experience you can see that this total velocity V total depends on the mass of the particle and also depends on the charge 1 by q B cell. So, different polarities of charges will experience drifts in different directions. So, this is the discussion about combining both the drifts which are curvature and gradient drifts. Thank you.