Plasma Physics and Applications

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Week – 04

Lecture 20: Gradient and Curvature Drifts

Hello dear students. We are discussing the gradient drift. The objective of gradient drift is to understand the plasma motion in a realistic magnetic field. What is the realistic magnetic field? Realistic magnetic field will never be uniform and will always be curved. It has to be curved to obey the Maxwell equations. We will continue our discussion.

We have derived an expression for the gradient drift which is V del B the gradient drift as minus plus V perpendicular RL by 2Bz dou Bz by dou y along x cap. We have understood what is the approximation that we have used for obtaining this expression. Now the lines of force in our last discussion were assumed to be coming out of the plane of the paper and we have also defined the gradient. Now let us see if the lines of force are not straight but they are curved.

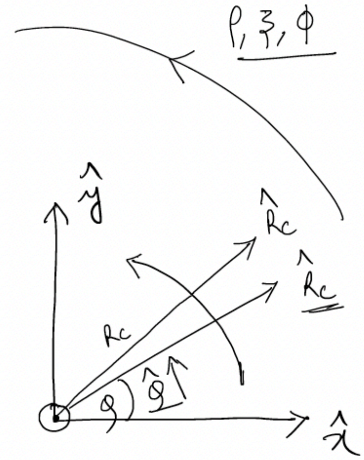
We have a line of force which is curved like this. What will happen in that case? If they are moving from north to south what will happen? How will the plasma motion be different in this situation? For this let us consider the magnetic field B parallel to the phi cap direction. We are looking at the cylindrical coordinates now. Let us say we have x cap and we have y cap and we have this z out of this plane. This is the radius Rc and we define phi cap in this direction or the angle itself as phi and the unit vector which is denoting the increase of the angle away from the x axis is phi cap.

This is the angle phi and the unit vector the corresponding unit vector is in this direction. The radius of curvature is if a magnetic field line is like this we can say that the magnetic field line is now parallel to the phi cap. The magnetic field line is going like this. Now since the magnetic field line is not a straight line rather it is curved the radius of curvature is denoted by this Rc cap. Rc cap is the unit vector.

$$V_{\nabla B} = \mp \frac{v_\perp r_L}{2B_z}\frac{\partial B_z}{\partial y}\hat{x}$$

let us consider $\vec{B} \parallel \hat{\phi}$

$$\vec{F} = q(\vec{V}\times\vec{B})$$



r, z, ϕ $\quad$ r-component equation

$$m(\ddot{r} - r\dot{\phi}^2) = -q(\vec{V}\times\vec{B})_r$$

$$V = (v_r, v_\phi, v_z)$$

$$B = (0, B_\phi, 0)$$

$$\vec{V}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ V_r & V_\phi & V_z \\ 0 & B_\phi & 0 \end{vmatrix}$$

The radius of curvature has to be defined with respect to the center of the coordinate system and you say this is Rc and the increasing direction is since this is Rc, Rc cap has to be this both of them are same actually. Now the configuration that we have taken is the magnetic field is parallel to the phi cap. What is phi cap? Phi cap denotes the direction in which the angle phi is increasing and what is angle phi? Phi is denoted in a counter clockwise direction away from the x cap. This is the frame that you have this is geometry that you have taken. The magnetic field line is parallel to the phi cap and the angle is measured away from the x axis and towards the y axis.

Now if you are taking the cylindrical coordinates we will write so our equation is still the same what is it? The force is equals to Q times V bar cross V bar is still the force. But we take a cylindrical coordinate system we write the respective equations as the we take R component, Z component and phi component these are 3. The R component equation so you can also write this as rho Z phi like that. The R component so if you take any basic mathematics textbook and refer to cylindrical coordinates you can get this equations. This is the derivative how they are written in different coordinate systems.

$$m(\ddot{r} - r\dot{\varphi}^2) = -q(-v_z B_\varphi)$$
$$= q v_z B_\varphi$$

$$m\ddot{r} = \underbrace{q v_z B_\varphi} + \underbrace{m r \dot{\varphi}^2}$$

Acceleration term

$$m \frac{dv_z}{dt} = -q v_r B_\varphi$$

$$\vec{F} = m r_c \dot{\varphi}^2 \hat{r}_c$$

$$V = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

So we will write m times r double dot minus r phi dot square is equals to minus Q times V cross B you will only take the R component the radial component. Similarly V is defined in V R, V phi, V Z the magnetic field B is now 0 B phi 0. So V cross B can be evaluated as i, j, k, V R, V phi, V Z, 0, B phi, 0. If you expand the cross product we can write the radial component the choice of unit vectors could be r cap, phi cap and z cap. After expanding the curl we can write m r double dot minus r phi dot square is equals to minus Q minus V Z, B phi or it is equals to Q, V Z, B phi.

m r double dot is equals to Q, V Z, B phi plus m r phi dot square. So r double dot is dimensionally equal to m a this is equals to the acceleration. So what you have is the momentum equation which is m a the force is equals to sum of the component due to the magnetic field and a component due to the magnetic field which is along phi cap. So this is the acceleration term. We understand this much.

What have we got? We have got a term which is because of the magnetic field and m r phi dot square and if you write the z components m d V Z by dt as minus Q V R, B phi. So because of the curvature of the magnetic field we have got an additional term. So inhomogeneity in the magnetic field now appears through B phi that is the basic idea. So B phi the additional term is basically due to the curvature. So you write the additional term just take out the additional term F is equals to m r c because we are looking at r c for a particular flux line which is along r c cap.

$$\vec{V}_{curv} = -\frac{m r_c \dot{\theta}^2 \hat{r}_c \times \vec{B}_\theta}{B^2 q}$$

$$\vec{V}_{curv} = \frac{m v_{\parallel}^2}{q r_c} \frac{\hat{r}_c \times \vec{B}}{B^2}$$

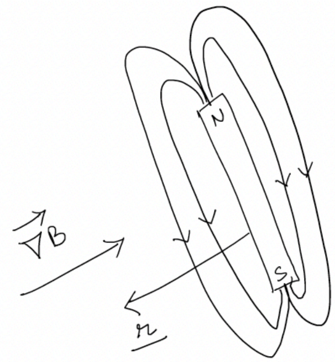
$$\boxed{\left.\begin{aligned} \vec{V}_{e_c} &= -\frac{m v_{\parallel}^2}{q r_c} \frac{\hat{r}_c \times \vec{B}}{B^2} \\ \vec{V}_{ion_c} &= \frac{m_i v_{\parallel}^2}{q r_c} \frac{\hat{r}_c \times \vec{B}}{B^2} \end{aligned}\right\}} \Longrightarrow \begin{aligned} &\propto \frac{1}{r_c} \\ &\propto \frac{1}{B} \end{aligned}$$

So V phi you should remember V phi is r phi dot. Now this is the additional force that the particle is experiencing. Now we can evaluate the curvature drift or we call this as from the generalized drift velocity expression. to use the generalized drift as 1 by Q times F cross B by B square. But now in order to accommodate the movement of the particle along a curved line of force we take this force and evaluate the drift pertaining to that.

So V we are going to call it as V curvature is equals to minus m r c phi dot square along r c cap times B phi divided by B square cube. Now this velocity V phi is nothing but r phi dot which can also be called as the parallel velocity because it is along the phi direction. So we can write V curvature is equals to m V parallel square by Q r c times r c cap times cross B by B square. This is V curvature. This is the curvature drift velocity.

$$\vec{V}_R = \frac{m v_\parallel^2}{q R_c^2 B^2} (\vec{R}_c \times \vec{B})$$

$$V_{\nabla B} \perp B, \nabla B$$



∇B lines of force are always curved

$$V_{\nabla B} = \mp \frac{v_\perp r_L}{2B^2} (\vec{B} \times \nabla \vec{B})$$

$$= \mp \frac{m v_\perp^2}{2q} \frac{(\vec{B} \times \nabla \vec{B})}{B^3} \qquad r_L = \frac{m v_\perp}{qB}$$

$$\frac{1}{2} m v_\perp^2 = W_\perp \quad (\text{K.E in } \perp^r \text{ direction})$$

$$\boxed{V_{\nabla B} = \mp \frac{W_\perp}{q} \frac{\vec{B} \times \vec{\nabla} B}{B^3}}$$

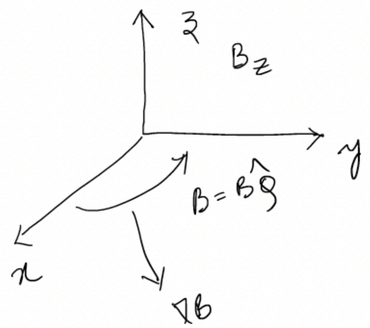
Now what you see from here is that the curvature drift velocity is dependent on the charge. So it will be in different directions for different polarities of the charge. And more importantly so we can write the electron curvature drift V ec as minus m V parallel square divided by Q r c times r c cap cross B by B square. The ion curvature drift has m i V parallel square by Q r c r c cap times B by B square. So from this we can infer that the curvature drift velocity is proportional to 1 by r c.

So greater the radius of curvature the drift velocity will be smaller and is proportional to 1 by B not B square 1 by B. So this is something about the curvature drift. Now what we have learned so far is we have learned that if there is a gradient in the magnetic field we have to think of the gradient drift velocity and if there is a curvature there is a curvature drift velocity. But most importantly we think of any magnetic field it is always accompanied by gradient and curvature at the same time. So an example is if you take a bar magnet like this you have the north and south pole.

The magnetic flux lines will converge and diverge from its poles. Now you see as you move radially away from the bar magnet the magnetic field strength will decrease that means as you go farther and farther its strength will decrease. So it is safe for us to assume that there is a gradient towards the magnet. And at the same time this magnetic field lines they have to converge and diverge at the poles that is the fundamental nature of the magnetic field. So if they do not converge or diverge that means you allow a possibility of magnetic monopoles to exist.

$$\vec{V}_{total} = \vec{V}_{\nabla B} + \vec{V}_R \qquad \downarrow$$

$$\vec{V}_{total} = \underbrace{\frac{1}{q}\,\frac{m v_{\parallel}^2}{R_c^2}\,\frac{\vec{R}_c \times \vec{B}}{B^2}}_{\text{Parallel component of velocity}} + \underbrace{\frac{m v_{\perp}^2}{2q}\,\frac{\vec{B} \times \vec{\nabla} B}{B^3}}_{\text{Perpendicular component of velocity}}$$



So if it has to converge it has to bend towards a magnetic pole. So this is the basic nature of the magnetic field that it will always have a gradient and its lines of force are always curved. Usually you take a very large distances over which the magnetic field can be assumed to be parallel but in totality it is always curved. So a realistic magnetic field always comes with these two basic properties that it will have a gradient and it will also have a curvature. Now we have evaluated expressions for the gradient drift as well as the curvature drift.

Now we have to find a mechanism wherein we can add these two drifts and give an expression which is valid for any magnetic field as long as it is not varying with respect to time. So we know that the curvature drift let us say we write Vr for curvature. We used Vc for curvature, Vc electron, Vc ion. We write a general expression without involving the charge Q to be positive or negative. We write m V perpendicular square by Q Rc square B square times Rc cross B.

So if you look at the earlier expression we have made use of the unit vector which is along Rc which is Rc by mod Rc. So this is the curvature drift. Now one thing is very clear, Vr is perpendicular to both Rc as well as B just like the V del B is perpendicular to the magnetic field itself and the gradient. So it will find a direction which is perpendicular to both of these directions at the same time. So as the particle is moving along the curvature it experiences this curvature drift and at the same time we have del B which is minus plus V perpendicular RL by 2B square times B cross del B.

So if you recall our last class we have used B cross del B by del y. The derivative of the magnetic field is now represented as a gradient in any possible direction which can be rewritten as minus plus m V perpendicular square by 2Q times B cross del B by B. What did I do? I have used RL is equals to m V perpendicular by QB at m V perpendicular square by 2Q B cube that is why this additional Q comes and multiplies this B square. Now we can write half m V perpendicular square as W perpendicular. What is W perpendicular? The kinetic energy in perpendicular direction.

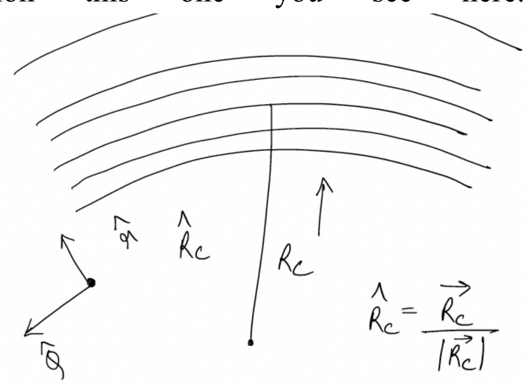
So we can write V del B as minus plus W perpendicular by Q times B cross del B by B cube. So what is the total drift the particle experiences? The total drift we can write the total drift is V del B plus V R. How can we write it? Total drift is equals to 1 by Q m V parallel square by RC square times. So this is the curvature drift which is taking care of the parallel component of the velocity because the particle is moving along phi direction and this drift seems to be affecting the phi component of the velocity which is the parallel component and this is affecting the perpendicular component of velocity. Now if you go back to this expression this one you see here.

$$B = (B_r, B_\varphi, B_z)$$

$$B_r = 0,\ B_z = 0,\ B_\varphi \neq 0$$

$$\vec{B} = B_\varphi(r)\,\hat{\varphi} \leftarrow$$

$$\vec{\nabla} B = \nabla B\,\hat{r} \leftarrow$$



$$\hat{R}_c = \frac{\vec{R}_c}{|\vec{R}_c|}$$

$$\vec{\nabla}\times\vec{B} = \left(\frac{1}{r}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\varphi) - \frac{\partial B_r}{\partial \varphi}\right)\hat{z}$$

So we basically have V phi when you have angular velocity that is how we write the angular velocity. So V phi is nothing but V parallel which is parallel to the magnetic field. So in your expression you have made use of this substitution only then otherwise your expressions were basically written in V phi. So this is the perpendicular and parallel components. Now we have to find a mechanism wherein we can combinedly write an expression involving both the gradient and the curvature drifts.

Now for this let us take a geometry. Let us say we have x, y, z we have the magnetic field along Bz and there is a curvature like this B is equals to B along phi cap and there is a gradient which is del B. So under this the particular geometry we write the magnetic field B as Br B phi Bz. So we have Br is equals to 0, Bz is equals to 0, B phi is not equals to 0 that means the curvature magnetic field we are trying to accommodate. So we have a situation like this in which the magnetic field lines are curved with a radius of curvature Rc and Rc cap is the interjector along this where Rc cap is written as Rc by mod Rc.

So if you look at this, so the coordinate system is you have this as the phi cap direction. Let us take a point. So this is along the curvature let us say this is along phi cap and this is R cap. So we can write B as B phi varying along R directed along phi. So as you move in the R direction in this direction you will encounter lesser and lesser number of flux lines but all of them are moving in the positive phi cap direction.

So this is the positive phi cap direction. So thus you have accommodated the provision of gradient as well as curvature. So this coordinate system is not actually representing the movement of the flux lines in the positive phi direction. I will try to redraw it.

So this is the direction. And you have a coordinate which is coming out of the plane which is the z direction the dot that I have written here. Now if this is del B how do you define del B? It has a magnitude del B and which is along R cap. So now I hope you have understood this. Let us go back and revisit the basic geometry of the magnetic field which is demonstrating the curvature as well as the gradient. You see here as you go farther and farther in this direction in the R C direction the number of flux lines is decreasing.

Let us say like this. That means there is a gradient in the magnetic field in the R direction that is what is indicated here. But what is the direction of magnetic field? The direction is along phi cap. Now if you are wondering what I am actually doing we have obtained expression for the total drift. The total drift is this. The first part is given by the curvature and the second part is given by the gradient.

Now if you look at it carefully we have obtained these two expressions individually by considering different geometries of magnetic field. While we were doing the gradient drift we never thought anything about the curvature. We thought it is a gradient which is present. We evaluated the average forces along all the directions then we use the generalized drift expression and obtain the gradient. We have this expression that is coming for that theory.

Then while we are looking at the curvature drift we did not thought of the gradient. We have taken a configuration in which the magnetic flux lines are moving in the positive phi direction and if the particle is travelling along that what is the force and what is the drift. Now what we are doing now is we are trying to see a complete picture wherein we are taking the cylindrical coordinates. We have assumed the gradient to be spread out in the r direction. If this is r direction in which the gradient is present that means as you go to higher values of r you are bound to find stronger magnetic fields and the magnetic field is now towards the phi cap direction.

Both of these defined clearly. This is the confusion. The magnetic field itself is along phi cap but the gradient is along r cap. Now let us evaluate the del cross B expression. In cylindrical coordinates we can write it as 1 by r dou B z by dou phi minus dou B phi by dou z along r cap plus dou B r by dou z minus dou B z by dou r along phi cap plus 1 by r dou by dou r of r B phi minus dou B r by dou phi along z cap. How did I get this? This is the expression for curl or del cross B in the cylindrical coordinates.

This is the general expression. We are going to substitute this configuration of the magnetic field into this and then we will try to see how we can get a combined expression of drift in a realistic magnetic field. This is how we write del cross B in the cylindrical coordinate system. We will try to make substitution of this geometry of magnetic field into this expression and try to see how we can write a single expression which involves the gradient as well as curvature drifts. Thank you.