Plasma Physics and Applications

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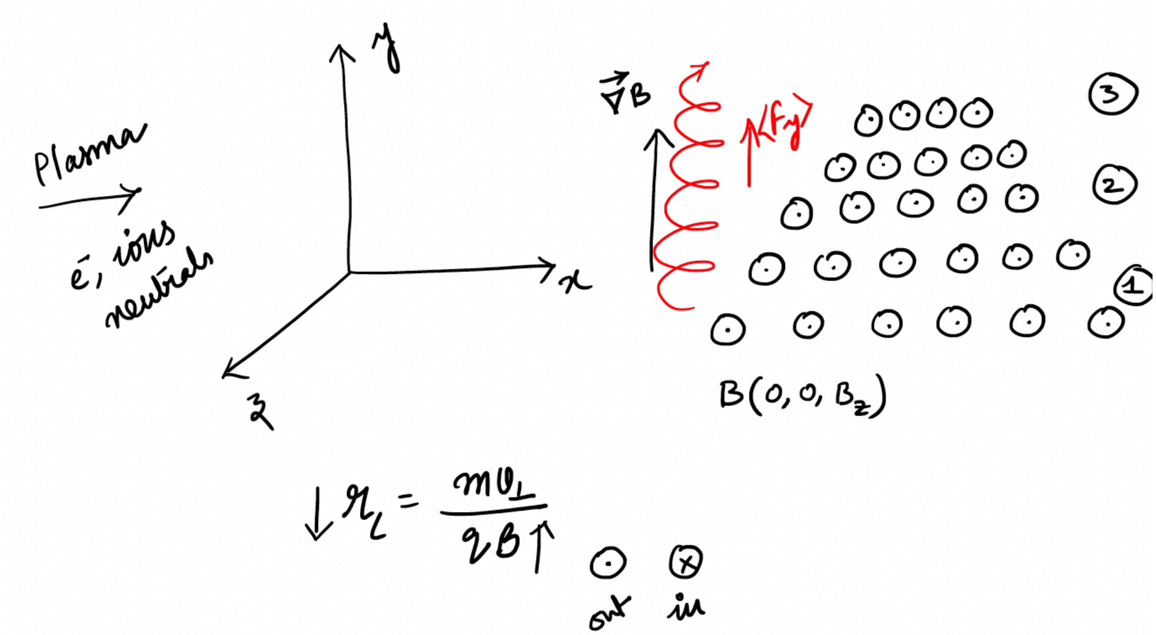
Week – 04

Lecture 19: Gradient Drift

Hello dear students. So in continuation to our discussion of plasma motion when it is treated like a particle, we have considered various configurations of electric and magnetic field primarily uniform and static. So, in today's class we will try to understand how the plasma will move or a charged particle will move when it encounters a magnetic field which is not uniform in space. So, this discussion is called as the gradient drift. So, what it means is that if you consider plasma as a composition of electrons and ions and if these electrons and ions are subjected to a magnetic field which is let us say B which is along a particular direction which is changing with respect to space. So, we are still retaining the variation of the magnetic field with respect to time as it is that means magnetic field is not changing with respect to time it is constant with respect to time but the magnetic field is let us say a particular direction is not 0.

What it means is that it is having a gradient or the amount of magnetic field that you can find at different points along a particular axis will be different or if you draw unit surface area the number of flux lines that you can expect to be crossing this unit area will be different at different places. So, which means that the magnetic field is having a gradient. So, you see this in this example that I have drawn see if you have taken this to be a unit surface area there are two magnetic flux lines which are passing and if you are going along this z direction you will see more flux lines are passing through this unit surface area. So, we can say that since the magnetic field is defined as the number of flux lines flux per unit area we can say that the flux is increasing or the magnetic field strength is increasing.

So, our motivation in studying this or in deriving this is going to be to be able to derive an expression for the trajectory of a particle or an expression for the generalized drift in a situation when the magnetic field is changing with respect to space. So, for this we have now invoked what is called as the magnetic field is no longer homogeneous it is inhomogeneous. So, the situation is inhomogeneity. So, inhomogeneity is defined to be in the length scale of L that means the magnetic field is changing over the length scale L or within the limit within the distance limit less than L the magnetic field can be assumed to be homogeneous. So, there are no changes in the magnetic field.



So, and we will make one more assumption saying that let us say if you have R L which is the radius of gyration of a particle in a magnetic field we can define R L to be m v perpendicular by q B. So, in order to match these two things what we will say is we will say that the R L the length scale over which one particle gyrates will be much in comparison to the capital L if L is the length of inhomogeneity. So, R L will be much less than L that means as long as the particle is gyrating in a single field orbit the magnetic field can be assumed to be a constant within that gyration length or the diameter. This is one important assumption that we make in order to proceed further. So, let us say the basic conditions are the magnetic field is inhomogeneous in space then we have it is static in time we can say just static let us say we say that it is static in time.

So, let us consider a coordinate system and we can with that reference we can see how the magnetic field can be described. So, this is your choice of coordinate system. So, you have x y and z and now you consider the magnetic field to be changing like this. So, let us say these are the magnetic flux lines. So, what I am drawing are the magnetic flux lines that are coming out of the page.

$$\vec{B} = B_z \hat{j} \; ; \; \vec{\nabla} B = \nabla B \hat{j} \qquad \vec{F} \Rightarrow V = \frac{\vec{F} \times \vec{B}}{qB^2}$$

$$\frac{\partial B_z}{\partial y} \approx \frac{B_z}{L} \Rightarrow \frac{B_z}{L} << \frac{B_z}{r_L}$$

$$\boxed{r_L \left(\frac{\partial B_z}{\partial y} \right) << B_z} \quad \text{—— } (1)$$

$$\vec{B} = B_0 + y \frac{\partial B}{\partial y} + \cdots\cdots \quad \text{—— } (2)$$

$$\vec{F} = e\vec{E} + e(\vec{v} \times \vec{B})$$

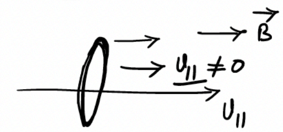
So, you always remember this reference or how the magnetic flux lines are denoted if you see across if you see a symbol like this you always remember that the magnetic flux lines are coming out of the plane of the surface. If you see something like this the magnetic flux lines are going in. So, one thing that is very clear from this figure is that the flux or the number of flux lines coming out of the surface per unit surface area is increasing in the positive y direction. So, this is the direction of the gradient del B. Gradient always indicates the direction in which the physical quantity is increasing.

So, this means that it is evident from the figure itself that the magnetic field is increasing as we go in this direction. So, let us put some references here let us say one reference one reference two reference three and we will denote the magnetic field itself to be in the z direction. So, the magnetic field is in the z direction its gradient is in the y direction I hope it is very clear. So, there is we do not talk about electric field as yet. So, this is the situation and if now plasma comes and experiences this magnetic field plasma which is assumed to be made up of electrons electrons ions neutrals.

Uniform magnetic field

$$(a)\begin{cases} x = r_L \sin \omega_c t \\ y = \pm r_L \cos \omega_c t \end{cases} \qquad \begin{matrix} v_x = v_\perp \cos \omega_c t \\ v_y = \pm v_\perp \sin \omega_c t \end{matrix} \Bigg\} (b)$$

$$\vec{F} = e(\vec{V} \times \vec{B})$$



$$V = (v_x, v_y, v_z)$$

$$B = (0, 0, B_z)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix} = \hat{i}(v_y B_z - 0) - \hat{j}(0 - v_x B_z) + \hat{k}(0)$$

$$\begin{cases} F_x = q v_y B_z \\ F_y = -q v_x B_z \\ F_z = 0 \end{cases} \ (c) \quad (a)(b)$$

But neutrals will not have any effect when they will not experience any effect when they see a magnetic field. Now, let us we have to understand how this magnetic field can influence the motion of electrons and ions. So, we can see that the magnetic field strength changes from 1 to 3. So, if you bring the expression for RL it is going to be m v perpendicular by q B. So, as the magnetic field strength increases the radius of gyration will decrease.

So, that means that as the particle moves from 1 to 3 its radius of gyration will start decreasing slowly. So, let us say how we can include all this. So, we assume the magnetic field to be B z along y cap and the gradient of magnetic field is the magnitude is del B which is across j cap. So, now as per the generalized drift, let us recall the generalized drift if you have the generalized force F which can give to a generalized drift which is F cross B by q B square. So, if the particle is acted upon by a generalized force which can be anything the resulting drift that the particle will experience is given by this expression.

Now, since we have considered a non uniform magnetic field, we will make an approximation which is suitable to make it convenient for the mathematical derivation. So, we have dou B z by dou y can be approximately equal to B z by L or what this approximation says is that the rate at which the magnetic field changes with respect to the distance along y is approximately equal to the ratio of magnetic field the strength of magnetic field over the distance L where L is the length scale of inhomogeneity. The length scale of inhomogeneity precisely defines the distance parameter over which the magnetic field changes. So, you can reasonably consider that before that length scale anything shorter than L any distance shorter than L the magnetic field can be assumed to be a constant. So, we are working at that approximation limit where we say that the magnetic field is changing yes, but it is not changing over this very small distance which we call as L.

$$f_x = q v_y B_z = q v_\perp \sin\omega_L t \left[B_0 + y \frac{\partial B_z}{\partial y}\right]$$

$$= q v_\perp \sin\omega_L t \left[B_0 \pm r_L \cos\omega_L t \frac{\partial B_z}{\partial y}\right]$$

$$f_y = -q v_x B_z$$

$$= -q v_\perp \cos\omega_L t \left[B_0 \pm r_L \cos\omega_L t \frac{\partial B_z}{\partial y}\right]$$

$$f_z = 0$$

Substituting (a) (b) & (2) in (c)

$$\vec{F} = (f_x, f_y, f_z)$$

$$0 \quad \underline{\langle f_x \rangle, \langle f_y \rangle, \langle f_z \rangle}$$

So, this implies that B z by L will be much less than B z by R L because we have used the fact that R L is much less than L. So, we can write R L times dou B z by dou y will be much less than B. So, this is a very important relation that we will need further. So, having established this relation between the length scale of inhomogeneity or the rate at which the magnetic field changes and with respect to the radius of gyration what we will we are going to do is we will. So, we have a limit within which the magnetic field is changing or with after which the magnetic field is changing.

So, what we will do is we will expand the magnetic field in that limit. So, magnetic field if you have let us say if you can write magnetic field is equals to is going to be B naught plus y dou B by dou y plus so on. What is this telling us? It is telling us the magnetic field can be obtained by a Taylor expansion around that length scale of inhomogeneity. So, we are going to neglect all these higher order terms because they will speak of a variation which is not actually present in the picture or which is very small and can thus be neglected. So, neglecting all the higher order terms we will take the magnetic field to be of this.

$$\langle f_x \rangle = q v_\perp \left[B_0 \langle \sin\omega_c t \rangle \pm r_L \langle \sin\omega_c t \cos\omega_c t \rangle \frac{\partial B_z}{\partial y} \right]$$

$$\langle \sin\omega_c t \rangle = 0 \qquad \langle \sin\omega_c t \cos\omega_c t \rangle = 0$$

0 to 2π

$$\boxed{\langle f_x \rangle = 0}$$

$$\langle f_y \rangle = \mp q v_\perp \left[B_0 \overbrace{\langle \cos\omega_c t \rangle}^{0} \pm r_L \underbrace{\langle \cos^2\omega_c t \rangle}_{1/2} \frac{\partial B_z}{\partial y} \right]$$

$$V = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

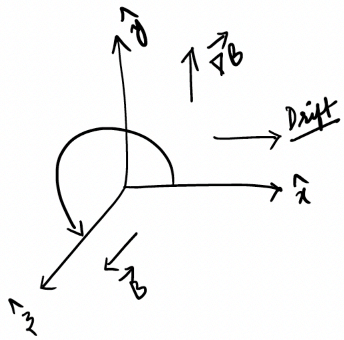
$$\boxed{\langle f_y \rangle = \mp \frac{q v_\perp r_L}{2} \frac{\partial B_z}{\partial y}}$$

$$\boxed{\langle f_z \rangle = 0}$$

And why are we taking this magnetic field? Because we are interested in evaluating this expression F is equals to E times E plus E times V cross V. Why is this expression important? So, we have already learnt the importance of this expression is the Lorentz force. So, this tells us the effect of the electric field and this tells us the effect of the magnetic field. But what we have known so far is that the magnetic field is going to be of this form because it is changing with respect to space. So, this term which is assumed to be 0 earlier is not 0 in this picture.

So, this is where we have to substitute this magnetic field and get some expressions. So, we will use a result which we already know. For example, we know in the case of a uniform magnetic field, what we know so far from the earlier discussion is that the position can be written as R L sin omega c t y is equals to plus minus R L cos omega c t. Whereas the V x, the x component of the velocity is V perpendicular cos omega c t V y, the y component of the velocity is plus minus V perpendicular sin omega c t. Now, the conclusion or the important aspect concept that you have to understand from this is that when the magnetic field is uniform, this is how the position coordinate x and y can be written as where R L is the radius of gyration, omega c is the gyration frequency, V perpendicular is the perpendicular component of velocity.

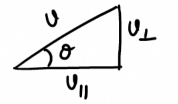
$$\langle F\rangle = \left\langle 0,\ \mp \frac{q v_\perp r_L}{2}\frac{\partial B_z}{\partial y},\ 0\right\rangle$$



$$\vec V = \frac{1}{q}\frac{\langle F_y\rangle \times \bar B}{B^2}$$

$$\vec V_{\nabla B} = \mp\frac{1}{q}\frac{\frac{q v_\perp r_L}{2}\frac{\partial B_z}{\partial y}\hat y \times B_z \hat z}{B_z^2}$$

$$\boxed{\vec V_{\nabla B} = \mp \frac{v_\perp r_L}{2 B_z}\frac{\partial B_z}{\partial y}\hat x}$$



$$v^2 = v_\perp^2 + v_\parallel^2$$

So, these two x and y exhibit a circular motion with a radius R L. So, if you take x square plus y square, you will get R L square. So, what does it mean? It is moving in a circular path with R L as the radius. Now, most importantly as long as there is no electric field, the particle will keep gyrating at a fixed position if it does not have any component of velocity which is parallel to the magnetic field. So, if you have V parallel and if this is the magnetic field, the magnetic field will make the particle move in this circle because the way it is drawn on this does not appear to be a circle, it is revolving in a circular motion.

So, if it has a parallel component of velocity which is non-zero, then the particle will keep moving in this direction and at the same time execute a circular motion. But if it is not there, it will just keep gyrating in a fixed orbit. So, now let us say we use this formula which is the force since there is no electric field, we will write E times V cross B. So, we have V the velocity written as Vx, Vy, Vz, the V as written as 0, 0, Bz. Evaluate this curl, what you can get is Fx is going to be Q Vy Bz.

So, if you want to know how I got this, it is straightforward actually minus Q Vx Bz Fz is equal to 0. How did I get this relation? This can be written from let us say we take the curl Ijk Vx Vy Vz 0, 0, Bz. What am I doing? I am just trying to evaluate the curl. So, if you break this, let us say I cap times this one Vy Vz minus 0 minus j cap 0 minus Vx Vz plus k cap times 0. So, this is the x component that I have written here, this is the y component I have written here, 0 is the z component.

So, I have written this just for the sake of clarity so that you may not think how did I get this relations. Now, the task is I have these expressions for Fx, Fy, Fz, we know what is Vy Vx from these expressions. Let us say we call this as, so this expression is an important expression, let us say we call this as 1, this as 2, this set of equations as A and this set of equations as B. So, we can use set of equations let us say B into this and we will rewrite as a matter of fact when we write A and B into this, we can rewrite it as Fx is equals to q Vy Bz which is equals to q V perpendicular sin omega c t since Bz has to be written as B0 plus y dou Bz by dou y. You see here, so Bz has to be written as per the expression 2.

So, what we have made use of is we have this set of equations as C. So, substituting equation B and 2 in C, we get this and which can be further simplified as q V perpendicular sin omega c t times B0 plus minus using for y RL cos omega c t dou Bz by dou y. And we have Fy as minus q Vx Vz. So, substituting similarly we can write minus q V perpendicular cos omega c t times B0 plus minus RL cos omega c t dou Bz by dou y Fz is 0. Now what I want you to understand is that generally when a particle moves in a magnetic field, it experiences force which changes its direction, only direction.

But here what you are able to see from these three expressions or the two expressions is that the particle is not just experiencing a force because of B0. It is also experiencing a force because of the way it changes with respect to the magnetic field, you see plus minus. Plus minus is there to accommodate the polarity of the charge, but otherwise one thing is clear. We have an additional term appearing in the force expression which conveys the message that the particle is experiencing additional force because of the gradient or change in the magnetic field. So, Fx, Fy, Fz are the magnitudes of the forces.

Now what we are interested to see is that if a particle is gyrating in one orbit, what will be the average amount of force that the particle experiences within one gyro orbit? So, if you write F the force as a non-zero force Fx component Fy Fz. What we are interested to see is what is the average amount of force the particle experiences in one gyro orbit? The moment it completes one gyro orbit, what is the average amount of force? So we have to evaluate if you want to find out the total average force, we have to evaluate what is the average force Fx, what is the average force Fy and what is the average force Fz. How do we characterize one gyro orbit in terms of distance or frequency anything like that? So let us say we write average force component Fx as Q V perpendicular B0 average sin omega ct plus minus RL sin omega ct cos omega ct dou Bz by dou y. You see this what I have done is I have taken sin omega ct into the bracket and written average of all the parameters. So, when you take an average of during one gyro orbit, how will the things change? We know from simple mathematics that average of sin omega ct is 0.

Average of sin omega ct between 0 to 2 pi, one gyro orbit 360 degree rotation is 0 and at the same time average of sin omega ct cos omega ct is also 0. So that means Fx average Fx is 0. What does it mean? It means the particle is not experiencing any average force in x direction over one gyro orbit. Let us see the story with Fy. If we take the parameters that are outside the bracket into the bracket, we can be able to write Q V perpendicular B0 cos omega ct plus minus RL cos square omega ct times dou Bz by dou y.

Average of this cos omega ct is 0 and average of cos square omega ct is half. So, Fy average is equals to minus Q V perpendicular RL by 2 dou Bz by dou y and Fy is this Fz is anyway 0. So, what have we realized? We have realized there is one non-zero component of the force along the y direction, the other components are 0. This is a very important result.

Let us just try to understand this. When the particle is moving in a magnetic field which is changing with respect to y, the particle will experience an additional force just because of the change of the magnetic field not because of the magnetic field but an additional force because of the change in the magnetic field. So, you see here Fy seems to be dependent on dou Bz by dou y and at the same time Fx is 0 and Fz is 0. So, if you take the original magnetic field configuration that we have started with, what you are able to see in this picture is that as the particle moves in this direction, it is experiencing a force in the y direction. So, Fy average Fy. So, what is the total effect of this? So, if you take the expression for the generalized drift which is 1 by Q F cross B by B square because of this additional force, we will write F is equal to 0.

Average force of the particle experiences within one gyro orbit is equal to Q V perpendicular R L by 2 dou Bz by dou y, 0. How did I get plus minus, minus plus? Because there is a plus minus here. So, when it comes out after this, I should have actually written plus minus instead of plus minus, I should have written minus plus because if this minus multiplies this plus minus and we have this minus plus. Now, let us evaluate the drift because this drift is because of the change in the magnetic field. We will say 1 by Q since Fy is the only non-zero component of the force Fy cross B by B square.

Since this is due to the gradient in the magnetic field, I will write gradient as this velocity as the gradient drift velocity is equal to 1 by Q minus plus 1 by Q Q V perpendicular R L by 2 dou Bz by dou y along y cap because this is Fy cross Bz, the direction of Bz is along z cap but the gradient of B is along y cap Bz square. Or in simple terms, we can write it as minus plus V perpendicular R L by 2 Bz dou Bz by dou y, y cross z is x cap. V perpendicular R L by 2 Bz dou Bz by dou y along x cap. So, this is the expression for the gradient drift velocity. You see what do we have? We have a velocity which is appearing in the left hand side.

V perpendicular is the perpendicular component of the velocity. Have V, this is the perpendicular component, this is the parallel component, V square is equal to square root of V perpendicular square plus V parallel square. So, now the important point here is this velocity is along x cap direction. It is not in the z or it is not in the y.

Let us put some reference here. The reference is you have this, this is the z cap and this is the x cap and this is the y cap. The choice of coordinates is that we are taking like this x, y and z. Now this is the direction in which the magnetic fields gradient is there and this is the direction the magnetic field is present. Now the drift is perpendicular to both these and this is the direction of the drift.

This is what you have to understand. Of course this is very clear since the beginning because we have a cross product and we can only expect the gradient drift to be perpendicular to both the physical parameters which you are involving in the curl operation. So, this is where we stop. So, this is an expression for the gradient drift velocity. Thank you.