Plasma Physics and Applications

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Week – 04

Lecture 18: Motion in Perpendicular Electric and Magnetic fields - II

Hello dear students, in today's lecture, we will try to understand what is the difference between the two particles. What is E cross B drift and what is a generalized drift and how gravity can also attribute a drift velocity to the charged particles. So, in the last lecture, we have seen this equation which is d square Vy by dt square is minus plus omega c d Vx by dt. We are trying to understand how a particle will move in the presence of perpendicular electric and magnetic fields. This will be a combination of particles movement due to acceleration by the electric field and that moment affected by a perpendicular magnetic field, both of them put together. We know that d square Vy has Vx on the right hand side.

We can use d Vx by dt from the earlier equations and we can write d square Vy by dt square as minus plus omega c is the gyration frequency times Q by m Ex plus minus omega c Vy. Simplifying it further, we can write it as d square Vy by dt square is minus plus omega c Q by m omega c is QB by m. So, that can be written as plus minus omega c Ex by Bz plus minus omega c Vy. We have used omega c is equal to plus minus QB by m.

We can simplify it as d square Vy by dt square is plus minus omega c goes inside plus minus into minus plus will become minus which is minus omega c square Ex by Bz minus omega c square Vy or we can write it as d square Vy by dt square is minus omega c square times Ex by Bz plus Vy. Let us say we call this equation as equation number 1. What you see here? This is a ratio which is equal to the dimensions of velocity. This is the velocity of speed of light plus the velocity. So, this is a constant here and this is a time dependent quantity which appears to the right.

Now, in order to simplify this equation, what we do is, let us say we can write Ex by Bz plus Vy, let us call it as x. So, if that is the case, then taking a double derivative of this will become d square Vy by dt square is d square x by dt square. d square Vy by dt square is equal to d square x by dt square. So, we know from the earlier equation as if you write this to be d square Vy by dt square is equal to minus omega c square x. What have I done? If I call this everything that appears within this bracket as x, we can write this as d square Vy by dt square is equal to minus omega c square x and by taking a double derivative of this equation, we got d square Vy by dt square is equal to d square x by dt square because this is a constant, a derivative of this will become simply 0.

$$\frac{d^2 v_y}{dt^2} = \mp \omega_c \frac{dv_x}{dt}$$

$$\frac{d^2 v_y}{dt^2} = \mp \omega_c \left[\frac{q}{m} E_x \pm \omega_c v_y \right]$$

$$\omega_c = \pm \frac{qB}{m}$$

$$\frac{d^2 v_y}{dt^2} = \mp \omega_c \left[\pm \omega_c \frac{E_x}{B_z} \pm \omega_c v_y \right]$$

$$\frac{d^2 v_y}{dt^2} = \left[-\frac{\omega_c^2 E_x}{B_z} - \omega_c^2 v_y \right]$$

$$\frac{d^2 v_y}{dt^2} = -\omega_c^2 \left[\underset{\uparrow\, c}{\frac{E_x}{B_z}} + \underset{\uparrow\, (t)}{v_y} \right] \quad \text{--- (1)}$$

Now, this one is actually equal to minus omega c square x. So, in turn, we have an equation d square x by dt square is equal to minus omega c square x. Now, we know x is equal to x by Bz plus Vy. So, in turn, we can write it as d square by dt square of Ex by Bz plus Vy is minus omega c square Ex by Bz plus Vy. So, if you are finding it uncomfortable to follow the flow of these equations, you can simply think like this.

This equation is there as it is since Ex by Bz is just a constant, I am just adding a constant on to the left hand side because the double derivative will make sure that it will be 0 anyway. So, this is the equation that we have got. Now, if I write this equation, let us say we have to solve this differential equation. So, in differential equation, we can write it as d square is equal to minus omega c square. That means, capital D is plus minus i omega c.

So, auxiliary equation has imaginary roots. So, we have to find out what will be the solution when the roots are imaginary, when the roots of this equation which are let us say we call it as R1 and R2 are imaginary or let us say complex. So, what are the roots? Plus minus i omega c. So, when you have this complex roots, the solution can be of this form C1 e to the power of R1 t plus C2 e to the power of minus R1 t something like that. So, we can use the well known methods in differential equation to solve this.

$$\frac{E_x}{B_z} + v_y = x \Longrightarrow \frac{d^2 v_y}{dt^2} = -\omega_c^2 x$$

$$\frac{d^2 v_y}{dt^2} = \frac{d^2 x}{dt^2} = -\omega_c^2 x$$

$$\frac{d^2 x}{dt^2} = -\omega_c^2 x \qquad x = \frac{E_x}{B_z} + v_y$$

$$\boxed{\frac{d^2}{dt^2}\left[\frac{E_x}{B_z} + v_y\right] = -\omega_c^2\left[\frac{E_x}{B_z} + v_y\right]}$$

$$D^2 = -\omega_c^2$$

$$D = \pm i\omega_c$$

So, if this complex roots are of the form alpha plus i beta, alpha plus minus i beta, then the solution would be Y of t. This is just a general method that I am describing. The solution would be a constant C1 times e to the power of alpha plus i beta for one root times t and so on. So, this is the method that we have to adopt when we are dealing with an auxiliary equation which has imaginary roots. So, here in our case the roots become plus minus i omega c.

So, that means X of t because we are seeking solution for X of t, X of t becomes e to the power of plus minus i omega ct multiplied by a constant let us say C1. So, we know what is X of t? We can write it as Ex by Bz plus Vy is equals to plus minus i alpha e to the power of i omega ct. Alpha is a constant actually. We can prove that alpha is actually equal to the V perpendicular velocity. So, which means how did we get this?

We got this from the auxiliary equation because the generalized solution for an auxiliary equation with imaginary roots will look something like this.

Roots r_1, r_2 are imaginary (complex)

$$\pm i\omega_c \qquad c_1 e^{r_1 t} + c_2 e^{-r_1 t}$$

$$\alpha \pm i\beta$$

$$y(t) = c_1 e^{(\alpha + i\beta)t}$$

$$\pm i\omega_c$$

$$x(t) = e^{\pm i\omega_c t} \; C_1$$

$$v_x, v_y, v_z$$

$$\left[\frac{E_x}{B_z} + v_y\right] = \pm i\alpha e^{i\omega_c t}$$

$$\Rightarrow \boxed{v_y = \pm i\alpha e^{i\omega_c t} - \frac{E_x}{B_z}}$$

$$\frac{dv_y}{dt} = \pm i(i\omega_c)\alpha e^{i\omega_c t}$$

So, Vy becomes plus minus i alpha e to the power of i omega ct minus Ex by Bz. So, this is one solution. We have the system of equations in three differential equations, three second order differential equations in Vx, Vy and Vz. If we solve these three equations, we will get the solutions which will describe how the particle is moving as a function of electric and magnetic fields. Now, we have to find out what is X? Let us say we take d Vy by dt will become plus minus i times i omega c just a derivative of this times alpha e to the power of i omega ct minus 0.

Multiplying d Vy by dt becomes minus plus omega c i square is minus. So, that multiplies with plus minus gives you a minus plus omega c alpha e to the power of i omega ct. Now, from the earlier equations which were taught in the last class, we know that d Vy by dt is equal to minus plus omega c Vx. This is the equation we started in this class. Comparing these two equations, we can write Vx is equal to 1 divided by minus plus omega c d Vy by dt.

So, Vx is equal to 1 by plus minus omega c into minus plus omega c alpha e to the power of i omega ct. Vx is equal to alpha e to the power of i omega ct. I could have just written this just by comparing. But anyway Vx is equal to alpha e to the power of i omega ct. So, now we have Vx, we have Vy.

$$\frac{dv_y}{dt} = \mp \omega_c \alpha e^{i\omega_c t}$$

$$\frac{dv_y}{dt} = \mp \omega_c v_x$$

$$v_x = \frac{1}{\mp \omega_c} \frac{dv_y}{dt}$$

$$v_x = \frac{1}{\cancel{\mp \omega_c}} \cancel{\mp \omega_c} \alpha e^{i\omega_c t}$$

$$\boxed{v_x = \alpha e^{i\omega_c t}}$$

$$\frac{dv_z}{dt} = \frac{q}{m} E_z$$

$$\boxed{v_z = v_{0z} + \frac{q}{m} E_z t}$$

We will need what is Vz? Vz is the uninterrupted part of the velocity because it is parallel to the magnetic field. The equation is d Vz by dt is equal to Q by m dz. So, Vz is equal to V0z plus Q by m dz. So, this equation tells you that there is a uniform acceleration along the z direction. So, what we have is, let us write them together.

Vx is alpha e i omega ct Vy is plus minus i alpha e to the power of i omega ct minus Vx by d Vz is equal to V0z plus Q by m dz dt. So, if you compare these three equations with Vx, Vy, Vz in the case of a isolated uniform static magnetic field alone, we will realize that what has changed. The only thing that has actually changed is the additional factor Vy has this additional factor of Ex by Bz. So, this factor is actually a constant. The ratio is a constant.

The ratio is independent of time and this additional factor is actually referred to as drift. So this velocity is not changing with respect to time. Vy this part is changing with respect to time. So, that means you have a term which is time dependent and another term which is independent. So, when you take a derivative of Vy, you will realize that only the time dependent term will sustain, the other term will become 0.

That means that you have a constant velocity added to the Vy component. So, how do we understand this? In order to easily understand this, we can take the force is Q times V cross B plus E. So this one, the constant part which we refer to as the drift velocity can be considered as the complementary solution of the differential equation. And this complementary solution seems to be having some significance because this term came as an additional factor when you compare this scenario with an isolated magnetic field. So, isolated magnetic field has a simple situation where the particle will gyrate along a particular direction depending on the direction of magnetic field.

$$v_x = \alpha e^{i\omega_c t}$$

$$v_y = \pm i\alpha e^{i\omega_c t} - \frac{E_x}{B}$$

$$v_z = v_{0z} + \frac{q}{m} E_z t$$

$$v_y \left(\frac{E_x}{B_z} \right)$$

<u>Drift</u>

So, the velocity component along the magnetic field will remain a constant, it will not change but the velocity components Vx and Vy which are perpendicular to the direction of magnetic field will execute a circular motion thereby with a non-zero z component what you will see is you will see a helical path along the direction of magnetic field but that is guaranteed. So, even if you take the first term on the right hand side of Vx and Vy, you will realize that these two terms combinedly will give you a sinusoidal motion or circular motion. But in addition what we have is Ex by B which is the complementary solution of the differential equation. So, now we have to understand what is this complementary solution? So, we will take it away and we will try to see how this is working. Now since the rate of change of velocity of this complementary part is 0, so we can write there is no net force acting along the direction of magnetic field.

So, we can write E plus V cross B is equal to 0. So, we will write the electric field is equal to minus V2 cross B. I am referring V2 because this is the velocity which has come into picture additionally. So, I will take a cross product from the right of this equation E cross B is equal to minus V2 cross B cross B. So, we have to use the formula of triple vector product A cross B cross C is equal to B.

I will write it here A cross B cross C is equal to B times C dot A minus A times C dot B. So, if I use this here on the right hand side, I will get V2 times B square minus B times B dot V2. So, E cross B is equal to V2 B square minus B. Let us look at this equation and try to figure out what are the terms which are relevant. Left hand side E cross B, this term whatever will be the resultant of E cross B will be in a direction perpendicular to E as well as B.

So, that is one direction that you have to figure out. So, there is some velocity component which is in the perpendicular direction to E as well as B. If you look at these two terms along on the right hand side, you have this term which is V2 B square this is along V2. But this one B dot V2 the dot product, the resultant of this dot product on the right hand side B dot V2 will be a scalar. What will be the direction of this the entire term on the right hand side? This one after minus this will obviously be in the direction of B.

$$\vec{F} = q(\vec{V}\times\vec{B}+\vec{E})$$

$$\vec{E}+\vec{V}\times\vec{B}=0$$

$$\vec{E} = -(\vec{V}_2\times\vec{B})$$

$$\vec{E}\times\vec{B} = -(\vec{V}_2\times\vec{B})\times(\vec{B})$$

$$(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{C}\cdot\vec{A}) - \vec{A}(\vec{C}\cdot\vec{B})$$

$$\vec{V}_2(B^2) - \vec{B}(\vec{B}\cdot\vec{V}_2)$$

$$\boxed{(\vec{E}\times\vec{B}) = \vec{V}_2 B^2 - \cancel{\vec{B}(\vec{B}\cdot\vec{V}_2)}}$$

1 2 3

Now, what we can say comfortably is that the resultant of E cross B will be in a direction perpendicular to E and B. So, it will not be in the direction of E or B. But this term let us say we call it as 1, 2, 3. The third term will be along B. So, if you try to account only those terms which are in the direction of E cross B or which can be in the direction of E cross B, we can neglect this term and say that E cross B is equals to V2 B square.

So, like I said before V2 is actually referring to the complementary part of the solution of the differential equation which is Ex by B which is the time independent part. The first one is time dependent and this is the complementary part. So, the complementary part if you take out it will look something like this V2 is E cross B by B square or this velocity we call it as Vd the drift velocity is across of E and B divided by B square. So, this is the concept of the E cross B drift velocity. What have we understood? We have understood three things here.

When we have perpendicular E and B fields both these fields will combinedly act on the particle. Second thing the Vx Vy will continue to give you circular motion. Third one the complementary solution of the differential equation will give you an additional term which you call as the drift velocity and this drift velocity will be in a direction perpendicular to E as well as B. So, this is the conclusion. This additional term or this velocity is called as the drift velocity.

Now what is the meaning of this drift velocity? It is a very difficult or complex concept. When we talk about drift velocity we generally referred to the movement of the guiding centers. Let us say if this is the trajectory of the particle. Imagine a center of the circle this point is called as the guiding center. I will explain the concept of drift in a simple way.

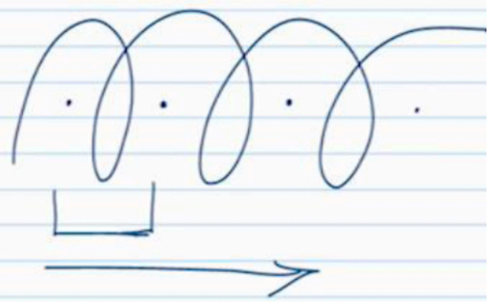
In the absence of any external electric field what you can expect is that or when only the magnetic field is gyrating the particle and the particle has a non-zero component of velocity along the direction of magnetic field what will happen? It will continue to execute circular motions with equal pitch. Pitch is this distance between two successive positions of the guiding center of two circular loops. But when you have an electric field along this direction what electric field will do is electric field will accelerate. You see this one.

So, Vz is equal to V0z plus Q by mEz into T. So, what it means is that V0z is the initial velocity of the particle plus a linear term is there. What it means is that with time the particle will move in such a way that the successive pitch between the position of the particle will keep increasing because of the acceleration that is brought into the picture by the electric field along the direction of the movement. So, this is the drift of the guiding centers. So, how the guiding center moves with respect to time is given by this drift. So, this drift is a very important concept E cross B or the configuration of perpendicular electric and magnetic fields will give you the first idea of drift.

$(\vec{E} \times \vec{B}) = V_2 B^2$

$$V_2 = \frac{\vec{E} \times \vec{B}}{B^2}$$

$$\boxed{\vec{V}_D = \frac{\vec{E} \times \vec{B}}{B^2}}$$



1) $E \perp B$
2) $V_x, V_y \Rightarrow$
3) Complex $\Rightarrow \underline{V_D}$

We will continue discussing about various different types of drift velocities as we change the configuration of electric and magnetic fields which we will discuss in the next lecture. Thank you very much.