Plasma Physics and Applications

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Week – 03

Lecture 15: Single-Particle Motion in Uniform Magnetic Field -III

Hello dear students. In today's lecture we will continue our discussion on understanding the particle motion in an isolated uniform and static magnetic field. In the last lecture we have derived v x, v y and v z. So, v x is v x 0 cos omega t plus v y 0 sin omega t, v y is minus v x 0 sin omega t plus v y 0 cos omega t, v y is minus v x 0 sin omega t plus v y 0 cos omega t and v z is v z 0. The meaning of these relations is that v x and v y will continue to decide the perpendicular component of the velocity and v z will remain a constant under the influence of magnetic field. This is pretty much known to us.

So, we have the magnetic field and v perpendicular and v parallel. V perpendicular results from v x and v y, it is a v x square, v x square plus v y square will be equal to v perpendicular and v parallel will be the velocity which is along the magnetic field. So, the magnetic field is v z. Now, if we combine we have already done this v x square, v x square plus v y square turns out to be v x 0 square plus v y 0 square.

$$v_x^2 + v_y^2 = v_{x0}^2 + v_{y0}^2 = v_\perp^2$$

$$\underline{\Omega}$$

$$\begin{cases} v_x = v_{x0}\cos\Omega t + v_{y0}\sin\Omega t \\ v_y = -v_{x0}\sin\Omega t + v_{y0}\cos\Omega t \end{cases}$$

$$v_z = v_{z0}$$

$$x(t) = \int v_x \, dt$$

$$\left[x = \frac{v_{x0}\sin\Omega t}{\Omega} + \frac{v_{y0}(-\cos\Omega t)}{\Omega} \right]^2$$

$$\left[y = \frac{v_{x0}\cos\Omega t}{\Omega} + \frac{v_{y0}\sin\Omega t}{\Omega} \right]^2$$

$$\underline{x^2 + y^2}$$

$$+$$

B_z

$v_\perp \quad v_\parallel$

$\downarrow$

$v_x, v_y \qquad \underline{v_z}$

$$v_x^2 + v_y^2 = v_\perp$$

We can call this as the v perpendicular square. Now, the velocity is resembling an equation of circle. So, the motion is circular motion in reality. We also realized what is the angular frequency that is associated with this circular motion. Let us go ahead step and find out how will the position will look like.

Let us say if I have to write x of t, then I have to integrate this velocity. So, we have to integrate v x d t which will be something like this x is equals to v x 0 sin omega t divided by omega plus v y 0 minus cos omega t divided by omega. This is x and similarly we can write as v x 0 cos omega t divided by omega plus v y 0 sin omega t divided by omega. So, there are some constants. So, I am neglecting all the constants.

So, x and y will look like this. Now, more importantly so what can we do about x square plus y square? Ultimately, this will be the trajectory of the particle x square plus y square will tell you how the particle is moving in the x y z plane. So, we can find out x square plus y square just squaring and adding these two things. So, I will write it anyway x square plus y square is v 0 x square by omega square. So, I have been calling v x 0.

$$x^2+y^2 = \frac{v_{x0}^2}{\Omega^2}\left[\sin^2\Omega t + \cos^2\Omega t\right] + \frac{v_{y0}^2}{\Omega^2}\left[\sin^2\Omega t + \cos^2\Omega t\right] + \cancel{\text{Remaining terms}} \rightarrow 0$$

$$x^2+y^2 = \frac{v_{x0}^2 + v_{y0}^2}{\Omega^2}$$

$$\boxed{x^2+y^2 = \frac{v_\perp^2}{\Omega^2}} \qquad x^2+y^2 = r^2$$

$$r_L = \frac{v_\perp}{\Omega}$$

$$\Omega = \frac{qB}{m} \Rightarrow \boxed{r_L = \frac{m v_\perp}{qB}}$$

So, let us say we write it as v x 0 to avoid any confusion and both the things are same. The way we have considered it at least sin square omega t plus cos square omega t plus v y 0 square by omega square times sin square omega t plus cos square omega t plus of course, the remaining terms. What are the remaining terms? You see what I have done is I have taken this expression squared it and added both. So, this is a plus b square a minus b square and this is a plus b square. So, adding all these terms we can realize easily that all the remaining other terms will get cancelled and we have x square plus y square is v x 0 square plus v y 0 square divided by omega square.

$$\Omega = \frac{qB}{m}$$

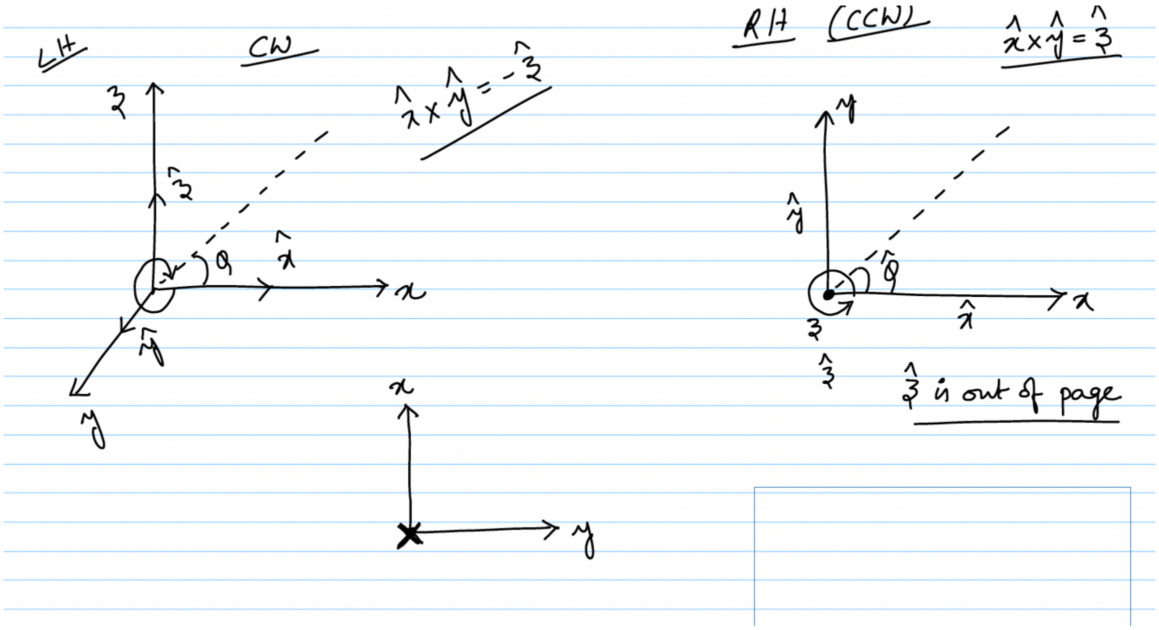
$$2f\pi = \frac{qB}{m}$$

$$T = \frac{2\pi}{\Omega} = \frac{2\pi m}{qB}$$

$$\boxed{\begin{array}{lll} \Omega = \dfrac{qB}{m} & & \text{rad/s} \\[2ex] T = \dfrac{2\pi m}{qB} & & \text{seconds} \\[2ex] r_L = \dfrac{m v_\perp}{qB} & & \text{m} \\[2ex] f = \dfrac{qB}{2\pi m} & & \text{Hz} \end{array}}$$

So, we can write x square plus y square as equal to v perpendicular square by omega square. Because v x 0 square plus v y 0 square is equal to v perpendicular square. Now, let us try to understand the meaning of this expression. This is the result of the entire discussion that we had. As the particle moves in a uniform magnetic field its position along x and y will be equal to ratio of the perpendicular component of velocity and the angular frequency.

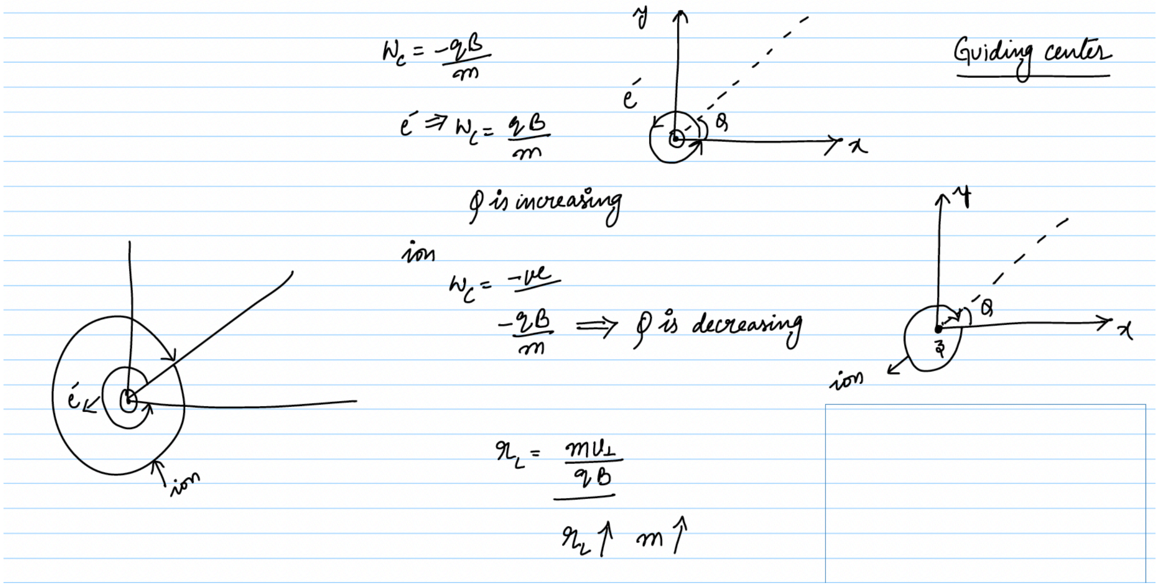
So, what kind of motion? This is an equation of circle actually. So, this resembles something like x square plus y square is equal to r square. That means this circular motion is executed by the particle with a radius of v perpendicular by omega. This distance v perpendicular velocity by angular frequency if you take that ratio you will realize that the dimensions are of the length. So, this is the radius of the gyration.



So, we call this as r L. What is this? This is the radius of gyration. Now, we know that the particles are executing circular motion. So, if you look at in this plane it will appear to be slanted but it is circular motion. Now, we have to understand we are yet to understand how will the particle let us say an electron move and an ion will move.

Will they move in the same direction? Will they move with the same velocity? Will they move with the same angular frequency? Will they move with the same radius and all these things? So, but for now we have established that the radius of gyration will be v perpendicular by omega. So, we know that omega is q B by m. So, using this we can write the radius of gyration as m v perpendicular by q B. The radius of gyration is m v perpendicular by q B. So, here we can understand few very important conclusions.

One, depending on the mass the radius of gyration will change and depending on the strength of the magnetic field also the radius of gyration will change. And looking at the frequency omega, omega is q B by m. So, we can write the frequency 2 pi f c is equals to q B by m. So, if we know the frequency we can know the time period is 2 pi the time period is 2 pi by omega which is 2 pi m by q B. So, most importantly omega is q B by m the time period is 2 pi m by q B.



And the radius of gyration is m v perpendicular by q B or if you want to write the frequency as it is q B by 2 pi m. So, the dimensions are rho omega which is radians per second for time period this is seconds radius meters and this is hertz. So, this is the conclusion of the entire discussion that we had so far. The particles motion is governed or characterized by these four parameters. Now, of course, the magnetic fields effect is realized only in the x and y components of velocity the z component of the velocity will remain a constant with respect to time.

So, v x and v y are referred to as the perpendicular components of the velocity and v z is the parallel component. Now, there is some more discussion that is related to this topic where we will try to establish how the particles trajectory can be drawn given the type of magnetic field. So, for that we have to understand the coordinate system or we have to be clear how we are defining the coordinate system and which particle will rotate in which particular direction. Let us say we have we define what is called as the right handed coordinate system. Right handed coordinate system you define it the easiest way to define it is let us say you have this x y z in the right handed coordinate system x y z are named such that they are following as per the curl of the fingers of your right hand.

$$\left\{\begin{aligned} v_x &= v_\perp \cos(\omega_c t + \psi) \\ v_y &= v_\perp \sin(\omega_c t + \psi) \\ v_z &= v_\parallel \end{aligned}\right.$$

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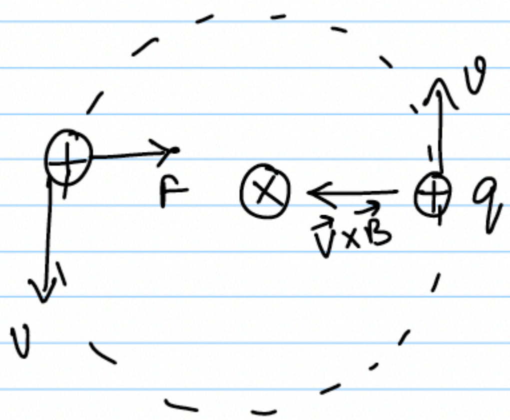
$$v_\perp = \sqrt{v_x^2 + v_y^2}$$



So, if you take your right hand so this is the way this is the counterclockwise direction C C W counterclockwise direction. So, you write the coordinates x y and z in the counterclockwise direction and what it means is you are writing x bar cross y bar is equals to z x cap cross y cap is equals to z cap. This is the rule that this coordinate system will follow. Now, the z coordinate is drawn like this but if you want to visualize this z coordinate z axis actually z axis is actually coming right out of the page. So, the best way to draw this is like this.

This is your x this is your y and this is your z this becomes x cap y cap and z cap. Now, you can choose your x to be anything else you can choose it to be in any other direction x is comfortably drawn generally in this way you can write x y z. Now, this is your plane and z axis is coming out of the page. So, like that you can choose this to be x this to be y and this to be z. So, the sense of rotation is important here.

So, this is the right handed coordinate system. So, exactly opposite will be valid for the left handed coordinate system. So, here z cap is out of page. What are we trying to understand? We are trying to we are in the process of drawing trajectories of electrons and ions when they are subjected to a magnetic field. Now, using this right hand thumb we can also indicate what will be the direction of the charged particles, but the sense of rotation the sense of gyration needs to be understood more thoroughly.

So, further we have to understand the coordinate system and in any given coordinate system what is the sense of rotation how does it correspond. Now, we define the angle phi always with respect to the positive x axis. So, this becomes phi cap or phi or this direction in which phi increases becomes phi cap the unit vector. The unit vector always denotes the direction in which it is increasing at the fastest. The left handed coordinate system can also be drawn easily.

We take this. So, in the left handed coordinate system the sense of x, y, z will follow the clockwise direction meaning x cap cross y cap will be equal to minus z cap. This is the difference. So, then I have to take this as x if I have to take this as x then I have to take this y and this is z and the unit vectors will be x cap, y cap, z cap. You see here the sense of rotation is like this and here the sense of rotation is like this. The easiest way to denote the left handed coordinate system is the left handed coordinate system.

If you choose your axis to be like this, this is your x. Now, here we cannot draw a dot it has to be like this. So, this is x, y, z. The z cap is going into the page that is the basic idea. Now, here again we define the phi with respect to the positive x axis.

So, you know how it is defined how the orientation is. So, just to conclude the right handed coordinate system indicates the assigning of axis x, y and z in the counter clockwise direction and the left handed system assigns in the clockwise direction. In the right handed system x cap cross y cap gives you z cap and in the left handed coordinate system x cap cross y cap gives you minus z cap. Once we establish this coordinate system, now we will make a generalized statement saying that if this coordinate system is what you have taken and if you have denoted the angle phi like this, then the electron will rotate like this, ion will rotate like this. All these things can be deduced by using the simple left hand or right hand rule, but then to be able to draw them we need to have a more thorough understanding.

Just draw the axis and then once you draw the axis you should be very confident to say that electron will of course, rotate like this and ion will rotate like this because they are circularly rotating each other. Now, let us say for reference from here onwards we will take the right handed coordinate system where you have x here, y here. So, we will take a right handed coordinate system in which the z axis is out of the page and we will assume that the magnetic field is along z axis. In this case, like I said the angle phi which quantifies the magnitude of rotation at any given point of time is represented with respect to the positive x axis. So, in that case we know that omega c is minus q b by m.

So, for an electron which is a negative charge particle omega c becomes positive which is q b by m. Now, remember this. So, if the charge is negative then the rotation will be such that phi is increasing. Just remember you draw this coordinate system and if it is a negative charge particle the rotation will be such that the value of phi is increasing. So, what I have drawn is the particle is rotating like this where the angle phi is increasing as the particle goes.

So, this is valid for an electron. That means increasing phi will be the sense of rotation for an electron. And if you take the exact opposite for an ion it will be omega c will be negative or it will be minus q b by m and an ion will rotate such that phi is decreasing. So, if I redraw the coordinate system this becomes x this is y and this is z. Now, the ion will exactly this is phi. So, the ion will rotate such that the angle phi is decreasing.

So, this will be like this. This is for the ion. So, the rotation will have a characteristic radius which is m v perpendicular by q b. Now, if you consider the two particles an electron and a singly ionized species atom something then you have the charges constant. So, the radius of gyration will be more when the mass of the particle is more. So, if you put that to perspective it will be something like this.

So, the electron will rotate like this. And the ion will rotate like this. So, the ion is having a larger radius in its circular motion. So, this is the right handed coordinate system the thumb is pointing the z axis. This is this curl is indicating how x y should be taken the CCW direction. Now, once we know this we can draw the trajectory of the particles accordingly.

Now, this is one more very important thing here as the particle is rotating around the center. So, this point around which the particle is executing this circular motion is called as the guiding center. And guiding center is very important in order to understand the particle drifts. So, now we can write v x as v perpendicular cos omega c t plus psi let us say v y is v perpendicular sin plus psi and v z is v parallel. Omega c t or omega c t omega c is nothing but the omega that we have been referring to which is the angle of frequency.

Now, this is the trajectory of the particle. What is psi? Psi is just an initial phase angle which defines the orientation of the particle when at t is equals to 0. So, that means if you have this the particle is at t is equals to 0 the particle is here and if it is an electron it will go like this if it is an ion it will go like this given that x and y are oriented in this direction. So, it will guide it like this electron will guide it exactly in the opposite direction. So, v perpendicular is square root of v x square plus v y square this follows directly from the equation. Now, in order to appreciate this equations of motion we can draw the trajectory and then we can realize how easily it represents.

Now, let us say the particle is rotating in a circular motion around we take the magnetic field to be going into the page then in that case the particle is rotating of course in a circular plane and if it is a positive ion you can apply this x y z coordinate system and figure out what is the sense of direction. So, this will be the velocity and the particle will always experience a force which is v cross B. You see the magnetic field is going in the velocity is in this direction then the other the magnetic field is in this direction the velocity is in this direction. So, you have the other force the Lorentzian force is exactly acting perpendicular to the velocity as well as the magnetic field.

So, this is how it is. So, the magnetic field is inside velocity is towards this and the force is in this direction. So, as the particle rotates always the velocity vector will be tangential to the trajectory the magnetic field will keep pointing towards the into the page and this thumb which is the direction of the force is always pointing towards the center. So, this is the Lorentzian force. Now, the particles charge is let us say q. So, at any point the force is acting inside the velocity you like this.

So, the particle is not falling into the magnetic field. So, there is a balance what is the balance the centrifugal force is balancing the Lorentzian force. So, m v square by r v perpendicular square by r is q times v cross B or you can simply write q v B because we have taken a perpendicular configuration. So, we can write r from here also I have indicated a minus to say that these two forces are opposite to each other m v perpendicular by q B. So, we have arrived at the same factor the radius of gyration is equals to m v perpendicular by q B this radius is called as the radius of gyration.

The radius of gyration. So, we will continue this discussion in the next lecture where we will try to understand some more aspects of this gyration. Thank you.