Plasma Physics and Applications

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Week – 03

Lecture 14: Single Particle motion in Uniform Magnetic Field - II

So, dear students, in today's lecture, we will continue our discussion on the single particle theory. The topic is a plasma particle under the influence of an isolated uniform and static magnetic field. So, the topic is to understand the particles motion in a uniform and static magnetic field. Now, in the last lecture, we have derived these two equations. So, we have considered the force is q times V cross V using this in the component form, we have got this equations m dVx by dt, m dVy by dt and m dVz by dt. So, the particle can have a three dimensional velocity and we have realized that the z component of the velocity which is parallel to the magnetic field will not be affected by the magnetic field strength.

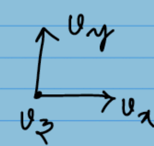
So, m dVx by dt is q Vy b is minus q Vx b and this is 0. So, the meaning of this is that the magnetic field will not be able to influence the z component of the velocity. So, you have Vx, Vy and Vz is pointing let us say the z direction is pointing in the out of the paper. Now, we are going to refer Vx and Vy as related to V perpendicular, the perpendicular component of velocity and Vz the parallel component of velocity.

Now, what we did in the last class is to couple these equations and we got two differential equations, two second order differential equations which are d square Vx by dt square plus omega c square Vx is equals to 0. And d square Vy by dt square plus omega c square Vy is equals to 0 and omega c is defined as minus q B by m and when we define minus q B by m, q can be positive or negative. So, q can be an electron in this case q is minus e or an ion where let us say q is let us say positive charge. Now, there is a specific meaning for this omega. What is this? Let us say if you just take these equations they represent a harmonic motion with a frequency omega c which is q B by m.

$F = q(\vec{v} \times \vec{B})$

— Uniform & Static Magnetic field

$$\left\{ \begin{aligned} m\frac{dv_x}{dt} &= q v_y B \\ m\frac{dv_y}{dt} &= -q v_x B \\ m\frac{dv_z}{dt} &= 0 \end{aligned} \right.$$



v_x, v_y $\quad v_\perp$

v_z $\quad v_\parallel$

$$\frac{d^2 v_z}{dt^2} = 0$$

$$\left\{ \begin{aligned} \frac{d^2 v_x}{dt^2} + \omega_c^2 v_x &= 0 \\ \frac{d^2 v_y}{dt^2} + \omega_c^2 v_y &= 0 \end{aligned} \right.$$

$$\boxed{\omega_c = \frac{-qB}{m}} \quad \begin{matrix} (-e) \\ (q) \end{matrix}$$

So, omega c is the angular frequency as per the differential equation we know it has to be in the units of angular frequency which is 2 pi nu or omega is defined in radians per second which is equals to minus q B by m. So, we call this frequency as the gyration frequency. Now, one thing is very clear these two equations in addition to we can also write d square Vz by dt square is equals to 0. These equations represent a harmonic motion or a periodic motion and we can intuitively say that the particle is executing a circular motion with a frequency omega c. Now, what is this frequency? Frequency tells you how many rotations does the particle do per unit time.

Now, if it is rotating circularly we have this omega c which can be positive or negative. So, for electrons omega c can be written as q B by m for electrons and omega c for ions can be written as minus q B by m. Now, the purpose of this negative symbol is that it denotes the direction of motion. For example, we take a coordinate system like this in this x, y and z. So, z is coming out of the plane.

$W_c = 2\pi\nu = -\frac{qB}{m}$ Gyration frequency

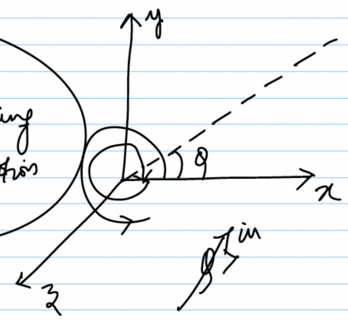
$W_c \rightarrow$ rad/s

$\vec{V}\times\vec{B}$

$\downarrow W_c = \frac{qB}{m\uparrow}$ for e^-

$W_c = -\frac{qB}{m}$ for ion

1) Direction of rotation
2) Radius of gyration
3) Any equation confirming the circular motion



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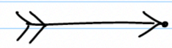

What kind of coordinate system is this? This is a right-handed coordinate system. Why do we say that? So, in a right-handed coordinate system, the z axis is coming out of the plane and the direction you take x, y, z is in this way. You take your right hand and your thumb is pointing outwards and the way the fingers are curling denotes the direction in which you should take x, y and z. And the magnetic field when it is pointing out of the paper you take it to be a dot and when it is going inside you take it to be a cross or plus. So, this is the arrow that we are seeing.

Let us say if it is coming out if the arrow is coming out the magnetic field direction is coming out you take this the point to be the indication of the outward magnetic field and if it is going in you see the tail of the arrow right. So, the tail of the arrow is symbolized with a cross. Now, that the point that I am trying to make here is now one thing is clear we are yet to establish the circular nature of the electrons orbit when it is subjected to the magnetic field. But we know even before the simple Lorentzian force will always try to act perpendicular to the velocity right. So, it is V cross B.

$$\left.\begin{aligned} \frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0 \\ \frac{d^2 v_y}{dt^2} + \omega_c^2 v_y = 0 \end{aligned}\right\} \quad \text{I}$$

$$m\frac{d\vec{v}}{dt} = q(\vec{V}\times\vec{B})$$

$$v_x = A\cos\Omega t + B\sin\Omega t \quad \text{— (1)}$$

$$@\ t=0 \Rightarrow v_x = v_{0x}$$

$$v_{0x} = A(1) + B(0) \Rightarrow \boxed{A = v_{0x}}$$

$$v_x = v_{0x}\cos\Omega t + B\sin\Omega t$$

$$\frac{dv_x}{dt} = (-A\sin\Omega t + B\cos\Omega t)\Omega \rightarrow (2)$$

$$m\frac{dv_x}{dt} = qv_yB$$

$$\frac{dv_x}{dt} = \frac{qv_yB}{m} \rightarrow (3)$$

So, the magnetic the force direction will be perpendicular to the direction of magnetic field as well as the velocity. And as the particle is deflected it will continuously act so that the particle will complete a circular path we know this very well right. Now, we are trying to establish the need of this negative symbol that appears right. The need is if we know this the coordinate system we should now know in which direction the particle will rotate. So, let us say for reference we define the we define Phi to be an angle with respect to the positive Z axis right.

Now, the increasing direction of Phi is increasing denotes one direction of rotation right. And the decreasing Phi let us say in another direction exactly opposite to this will indicate another direction of rotation. So, it is obvious at this point of time that both the part electron and ion will not rotate in the same direction they will rotate in opposite directions to each other. And since the mass of the ion is larger the frequency of the rotation will be smaller what does it mean? The number of rotations that the ion will make per second will be small in comparison to the number of electrons. So, in order to comment these are issues right now we have to make the we have to know the direction of rotation.

Compare (2) & (3)

$$\frac{q v_y B}{m} = \left[-A \sin \Omega t + B \cos \Omega t\right] \Omega$$

$$\cancel{\Omega} v_y = \left[-A \sin \Omega t + B \cos \Omega t\right] \cancel{\Omega}$$

$$v_y = -\underline{A} \sin \Omega t + \underline{B} \cos \Omega t$$

$$@ t=0 \Rightarrow v_{oy} = -A(0) + B(1)$$

$$\Rightarrow \boxed{B = v_{oy}} \; ; \; \boxed{A = v_{ox}}$$

$$v_x^2 = (v_{x0} \cos \Omega t + v_{y0} \sin \Omega t)^2$$

$$v_y = -v_{x0} \sin \Omega t + v_{y0} \cos \Omega t$$

$$v_z = v_{z0}$$

So, in which direction the electron will rotate within any given coordinate system left handed or right handed. If you give the coordinate system you should be able to tell in which direction the particle will rotate and then we have to know what will be the radius of gyration. The particles motion is referred to as gyration what will be the radius of gyration right. So, far we have not derived any expression for the radius of gyration we only have the expression for the frequency of gyration which is just inversely proportional to the mass. So, for the same charge and the magnetic field more the mass the heavier the particle it will rotate slowly and what about the radius? So, these two things we have to understand and we have to also understand the third thing is we have to derive an equation which conforms the circular motion right.

$$v_x^2+v_y^2$$

$$= v_{x0}^2\cos^2\Omega t + v_{y0}^2\sin^2\Omega t + \cancel{2v_{x0}v_{y0}\sin\Omega t\cos\Omega t}$$

$$+ v_{x0}^2\sin^2\Omega t + v_{y0}^2\cos^2\Omega t - \cancel{2v_{x0}v_{y0}\sin\Omega t\cos\Omega t}$$

$$\boxed{v_x^2+v_y^2 = v_{x0}^2 + v_{y0}^2}$$

$$x^2+y^2=r^2$$

Now ultimately we need to if we have to understand the problem completely we need to have answers for these three questions. Now so deriving this second order differential equations is not the complete story we have to solve them. Let us rewrite them here d square Vx by dt square plus omega c square Vx is equals to 0 and d square Vy by dt square plus omega c square Vy is equals to 0. Now at this point of time the perpendicular component of the velocity Vx and Vy is executing a circular motion what is the what is the parallel component doing? It is constant d square Vy by dt square is a constant right. Now let us take this equation as it is and try to solve them or try to obtain solution which will probably tell us more information about the nature of motion.

So, let us say for this type of equation by start with Vx, Vx is a cos omega t plus b sin omega t fine. So, we have these constants a and b right. So, let us say at t is equals to 0 at the beginning of this picture we take the velocity Vx to be equal to V0 x. When we substitute t is equals to 0 into this we will get V0 x is a times 1 cos of 0 is 1 plus b of b into 0 that implies a is V0 x. So, most importantly we are not saying b is 0 but the value of a would be equal to a0 x sorry V0 x the initial velocity along the x direction right.

So, we can now write in terms of Vx we can now write Vx is V0 x cos omega t plus b sin omega t. Now let us say we take dVx by dt is minus a sin omega t plus b cos omega t multiplied by omega. Fine now from the earlier equation we know that m dVx by dt is Q Vy b how did I get this? This is simply the f m d square x by dt square is equals to Q times V cross b. So, this is where we actually started anyway m dVx by dt or simply we will write m dVx by dt is Q times V cross b not Vx actually yeah V. Now the point is we are going to use this relation expanding the curl only gave us three equations by the way.

So, this is m dVx again I made a mistake m dV by dt it is m dV by dt is Q times V cross b right. So, this will give you m dVx by dt is equals to Q Vy b right. So, we also have dVx by dt equal to this. So, all of this divided by mass should or multiplied on the left hand side by mass should be equal to this right. So, both of the things are same or should

I write for more convenience we will write dVx by dt is Q Vy b by m fine.

Now let us say we call this equation this set of equations as rho 1 1 or this solution as equation number 1 and this one has equation number 2 and this one has equation number 3. Now what we can do is we can compare equation number 2 and both of them are pretty much the same. Then we can write Q Vy b by m is equals to minus a sin omega t plus b cos omega t multiplied by omega. Did I change anything? No we have dVx by dt all of this omega came out because we took a derivative of the velocity right fine minus a sin omega t plus b cos omega t multiplied by omega. So, dVx by dt is Q Vy b by m that is what I have written here right.

Now Q V by m as we know can be equal to omega, omega Vy is minus a sin omega t plus b cos omega t times omega. So, omega gets cancelled and we write Vy as minus a sin omega t plus b cos omega t. Now you just try to understand what I have done. I started by assuming a solution for Vx. What is Vx? Vx is the independent variable which appears in the second order differential equation right.

So, I assumed a solution for Vx and now this solution has two constants a and b. So, I do not know what are these values. So, what I can do is in order to find out I supply an initial condition saying that at t is equal to 0 the velocity along x would be called as an initial velocity V0 x fine. So, that gave me the value of one constant out of the two and then what I did I took a derivative of this velocity and then I compared it with the equation of motion m dVx by dt right. Now this is the way the two velocities are coupled.

So, if you want to find out the value of Vx the variation of Vx seems to be dependent on Vy. Similarly the variation of Vy with respect to time seems to be dependent on Vx. So, both of these things are there. So, this facilitates having an relation for Vx we will be able to find out what is Vy right. So, thus we have arrived at one point where we have actually removed or eliminated Vx out of this and we now have a relation for Vy.

See mathematically it is very simple right. So, Vy is equal to minus a sin omega t plus b cos omega t. So, we still have the same constant this is the beauty of the coupled Vx and Vy. So, now let us say at t is equal to 0 we will get Vy is equal to minus a into 0 sin 0 plus b into 1 which implies b is equal to V0 y at t is equal to 0 the velocity is V0 y. Now we have b is equal to V0 y and a is equal to V0 x right.

So, let us say we have and a is equal to V0 x. Now we are in a position to write Vx as well as Vy. So, then Vx is Vx0 that is a cos omega t plus Vy0 0 y or y0 it is okay both the things are same sin omega t. How about Vy? We also have Vy, Vy is equal to minus

Vx0 sin omega t plus Vy0 cos omega t.

So, we have both the relations. Now what about Vz? So, what do we have? We have m d square Vx sorry Vz by dt square is equal to 0 right this is what we have because there is no acceleration along the z axis right. Now if we integrate this equation once what we will get is m d Vz by dt is obviously a constant right. So, this constant it has to be a constant right. Let us call that constant as the initial velocity along the z axis. We have more reason to call that as the initial velocity because as per our understanding so far the initial velocity along the z axis will be uninfluenced by the magnetic field which means it will remain a constant of time.

So, we can write d Vz sorry Vz since Vz will remain a constant that is the point that we are trying to make right. So, Vz will simply be Vz0. We could have gotten rid of the mass here itself right it does not matter. So, m d Vz by dt is a constant or Vz will be Vz0 the velocity along the axis will remain a constant.

Now this is a constant. Now we have this Vx Vy and Vz. Now we have to understand the nature of the motion velocity x V x y and z. Let us take Vx square plus Vy square using the earlier relation we can simply write it as Vx0 square cos square omega t plus Vy0 square sin square omega t plus 2 Vx0 Vy0 sin omega t cos omega t plus Vx0 square sin square omega t plus Vy0 square cos square omega t minus 2 Vx0 Vy0 sin omega t cos omega t. See here go back Vx square is this a plus b square Vx0 square cos square omega t plus Vy0 square sin square omega t plus 2 Vx0 Vy0 cos omega t sin omega t and this one since we have a minus here we will get this minus. So, when we add these things so these two terms will get cancelled and sin square plus cos square omega t is 1.

So, Vx square plus Vy square is Vx0 square plus Vy0 square. So, this is a very important result. You see if you think it is of course implicit then the components Vx Vy Vz which are related by the curl operation are coupled with each other. Now when we decouple them we got a second order differential equation. This second order differential equations solutions when you add them what you when you square and add them seems to be related like this.

So, now what kind of motion is this Vx square plus Vy square is equals to Vx0 square plus Vy0 square. Now of course this is representing something like x square plus y square is equals to r square but now this is in velocity not in position. Now can we still understand something out of the velocities? Let us give it a try. So, in the next class we will extrapolate this derivation and find out how the position x and y will behave in the presence of an external magnetic field. Thank you.