Plasma Physics and Applications

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Week – 03

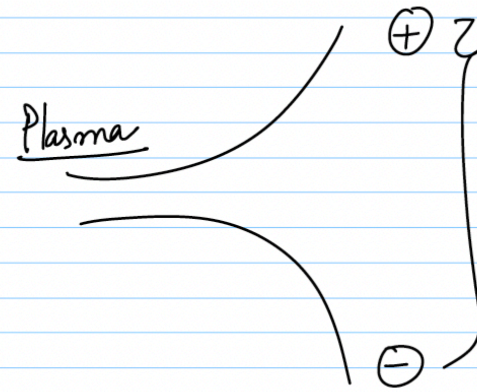
Lecture 13: Single particle motion in Uniform Magnetic Field

Hello dear students, we will continue our discussion of understanding a single particle in the presence of an electric field. We have considered an electric field to be static and uniform along the z direction and we got this expression in which Vx is a constant and Vy is a constant and Vz is Vz0 plus Q by m Ez times T. So, due to the application of electric field along the z direction, the velocity components along x and y directions will remain a constant but that means there is no acceleration along the x and y direction. But the velocity along the direction is changing linearly with respect to time by this factor. Now, the important conclusion that we have drawn in the last class is that depending on whether you have an electron where the charge is negative or an ion where the charge is positive, the particles will be separated. This electric field will move all the ions to one side and all the electrons to another side.

So, this may lead to the formation of a secondary electric field. So, plasma is coming from this direction. Now, let us try to understand the consequences of this. What we have is we have m dV by dT is Q times the electric field.

Let us take a dot product with V from the right. Let us say we will write m dV by dT dot V is Q E dot V. So, this is dV by dT dot V. So, we can write it as d by dT of half m V square. You take this derivative inside, V square is 2V dV by dT, 2, 2 gets cancelled, you still have the same form as on the left hand side.

$$v_x = K$$
$$v_y = K$$
$$v_z = v_{z0} + \frac{q}{m}E_z t$$



e^- ⊕

$q = -$ $q = +$

$$m\frac{d\vec{v}}{dt} = q\vec{E}$$

$$\vec{E} = -\vec{\nabla}\phi$$

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = q\vec{E}\cdot\vec{v}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q(-\nabla\phi)\cdot\vec{v}$$

So, we will write Q times E dot V. So, the electric field can always be represented as the negative potential gradient. Using this, we can write Q times E is minus del phi dot V. So, slightly modifying it, we can write it as d by dT of half m V square is equals to Q times the negative potential gradient times the velocity which is written as dR by dT, the rate of change of position or we can write d by dT of half m V square is equals to. So, this is the gradient, we will expand the gradient as, we will put minus here and I dou phi by dou x plus j cap dou phi by dou y plus k cap dou phi by dou z dot dR by dT i cap dou x by dou t plus j cap dou y by dou t plus k cap dou by dou t.

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q(-\vec{\nabla}\varphi)\cdot\frac{dr}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -q\left(\hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}\right)\cdot\left[\hat{i}\frac{\partial x}{\partial t} + \hat{j}\frac{\partial y}{\partial t} + \hat{k}\frac{\partial z}{\partial t}\right]$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -q\frac{d\varphi}{dt}$$

$$\boxed{\frac{d}{dt}\left(\frac{1}{2}mv^2 + q\varphi\right) = 0} \implies \underline{\text{Work done} = 0}$$

So, if you resolve this dot product on the right hand side, we will be able to write d by dT of half m V square is equals to, so i dot i is 1 from the simple vector calculus, you may be doing i dot i is equals to 1. Dou phi by dou x multiplied by dou x by dou t, you will get dou x dou x gets cancelled, you will get dou phi by dou t. So, we will have minus Q times d phi by dT or we can write d by dT of half m V square plus Q phi is equals to 0. What is the meaning of this? I hope you have understood how I got this. I have started with the basic Lorentz force.

Now, I am not trying to get an expression for the velocity, I have not touched the velocity on the left hand side, I let it remain as it is. But I have worked on the right hand side and have been able to prove that the rate of change of some of the kinetic energy of the particle and this is the electrical potential energy. This is the kinetic energy of the particle. So, the kinetic energy is by the virtue of the mass of the particle and its movement, the velocity. And Q phi, this one is by the virtue of the charge of the particle which is experiencing a potential phi which is because of the electric field.

Uniform Magnetic field

$$\vec{F} = q(\underbrace{\vec{E}}_{0} + \vec{V} \times \vec{B})$$

$$\vec{B}$$

$$\left\{ \frac{\partial \vec{B}}{\partial r} = 0, \quad \frac{\partial \vec{B}}{\partial t} = 0 \right.$$

Static & Uniform

$$\vec{E} = 0$$

$$m\frac{d\vec{V}}{dt} = q(\vec{V} \times \vec{B})$$

$$\vec{V} \cdot m\frac{d\vec{V}}{dt} = \vec{V} \cdot q(\vec{V} \times \vec{B})$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\left(\vec{V} \cdot (\vec{V} \times \vec{B})\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0}$$

So, the statement is the rate of change of total energy which is a sum of kinetic energy and the electric potential energy will remain 0. That means there will not be any change. So, particle although it is experiencing acceleration in the like we have seen in the last slide, see here the particle velocity is changing with respect to time. So, it is accelerating. But despite of that, the total energy of the particle will remain a constant.

So, can we go ahead and say that the net work done from the work energy theorem, we can say that the net work done is 0. Can we say that? Think about it. When there is no change in the energy, did we do any work by moving the particle from one point to another point? This is a more involved discussion. Let us try to understand or you try to think what is the inference on the work done when the total energy remains a constant, the sum of kinetic energy and electric potential energy is 0. When the particle is experiencing a static and uniform electric field along one direction.

$$v = v_{\perp} + v_{\parallel} \quad \text{——} \ (1)$$

$$\frac{dv_{\parallel}}{dt} + \frac{dv_{\perp}}{dt} = \frac{q}{m}\left[(v_{\parallel} + v_{\perp}) \times B\right] \qquad B_z$$

$$\frac{dv_{\parallel}}{dt} + \frac{dv_{\perp}}{dt} = \frac{q}{m}\left[\vec{v}_{\perp} \times \vec{B}\right] \qquad \frac{v_x, v_y}{v_z}$$

$$\frac{dv_{\parallel}}{dt} = 0 \Longrightarrow v_{\parallel} = k. \quad \text{——} \ (2) \qquad v_{\parallel} \times B = 0$$

$$\frac{dv_{\perp}}{dt} = \frac{q}{m}\left(\vec{v}_{\perp} \times \vec{B}\right) \quad \text{——} \ (3)$$

$$(4) \begin{cases} \vec{B} = B_z \hat{z} \\ \vec{B} = (0, 0, B_z) \end{cases}$$

Now, you must see we have simplified the case by considering the electric field only along one particular direction. Then we will move on to the next topic. Now that we have seen how the particles motion will be affected when it is subjected to an electric field. Now, do not think about particle rather you think about plasma. A weak collection of charged particles if it is subjected to the electric field, what will the electric field do? It will just separate the charges in the directions.

Now, let us think about the particle motion in a uniform magnetic field. You take a uniform magnetic field. What do you mean by uniform? Let us say we say that dou B by dou R, a generic position coordinate is 0. So, electromagnetic field is not changing with respect to space and we will also say that the magnetic field is static. For example, dou B by dou T is also 0.

$$\left.\begin{aligned} m\frac{dv_x}{dt} &= v_y B_z q \\ m\frac{dv_y}{dt} &= -v_x B_z q \\ m\frac{dv_z}{dt} &= 0 \end{aligned}\right\} \quad (5)$$

$$\vec{F} = q(\vec{v} \times B)$$

$$v = v_x, v_y, v_z$$

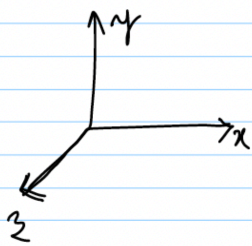
$$B = 0, 0, B_z$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix}$$

$$q(v \times B)$$

$$\frac{dv_y}{dt} = -\frac{q}{m} v_x B_z$$

$$\frac{dv_x}{dt} = \frac{q}{m} v_y B_z$$



So, these two conditions impose a static and uniform magnetic fields. This problem is of a more fundamental interest in plasma physics to be able to understand the particle trajectory in a magnetic field. So, when the particle is under the influence of a uniform electric field, the particle which has accelerated along the direction of electric field as long as it is a positive charge. So, the motion of particle or the charged particle under the influence of static and uniform magnetic field is of a more fundamental interest in plasma physics. So, we have considered the magnetic field configuration to be like this.

Now, let us say we will again bring the, we will also say that there is no electric field present. So, E is 0. So, where do we start? We always start at this equation, F is equals to Q times E plus V cross P. Since the electric field is now 0, the particle has some velocity because without velocity the force will be 0. The magnetic field cannot act on a particle which is at rest.

Despite the fact that it may have some charge, it cannot act until unless you have some velocity because this term has to be non-zero for the force to exist. There are two different ways of understanding this particle motion in uniform magnetic field. We will try to derive trajectories, we will try to see how the particles movement will look like in a three dimensional coordinate system. But we start here, so we say m dV by dt the force ma is Q times V cross P. And let us say we take a dot product with velocity from the left this time.

$$\left.\begin{aligned} &(a)\quad m\frac{d^2v_x}{dt^2} = qB\frac{dv_y}{dt} \\ &(b)\quad m\frac{d^2v_y}{dt^2} = -qB\frac{dv_x}{dt} \\ &(c)\quad m\frac{d^2v_z}{dt^2} = 0 \end{aligned}\right\} (6)$$

Using (5) in (6)

$$m\frac{d^2v_x}{dt^2} = qB_z\left(\frac{-qB_z}{m}\right)v_x$$

$$\frac{d^2v_x}{dt^2} = \frac{-q^2}{m^2}B_z^2\, v_x$$

$$\frac{d^2v_x}{dt^2} + \frac{q^2B_z^2}{m^2}v_x = 0$$

$$m\frac{d^2v_y}{dt^2} = -qB\left(\frac{qB}{m}\right)v_y$$

$$\downarrow$$

$$m\frac{d^2v_y}{dt^2} + \frac{q^2B_z^2}{m^2}v_y = 0$$

So, we take V dot m dV by dt is V dot Q times V cross P. So, we can write d by dt of half m V square is equals to Q times V dot V cross P. So, this V cross P term that appears here, the result of this cross product will be in such a direction which is perpendicular to the velocity as well as the magnetic field. But whatever it is, when you take a dot product of that quantity with V, with the velocity itself, it will be 0. It is a simple vector identity.

So, you have the dot product a dot b is mod a mod b cos theta. So, if the angle is 90, the dot product would be 0. Now, the result of V cross B will be a vector because it is a vector product. But it will be perpendicular to V. When you take this dot product, it will be 0.

This is a simple vector identity, we use it quite often in physics. So, this explanation was not necessary. So, let us say if the entire right hand side becomes 0, we will write d by dt of half m V square is 0. What does it mean? The rate of change of energy with respect to time becomes 0. So, the magnetic field is not able to do any work on the particle.

So, what happens? So, velocity is not changing, there is no acceleration in picture. As a result, the total energy remains a constant. So, now, the velocity being perpendicular to V cross B has made the right hand side 0. The static magnetic field cannot change the kinetic energy of the particle. It can only change the direction, we will come to that later.

$$\frac{d^2 v_x}{dt^2} + \frac{q^2 B_z^2}{m^2} v_x = 0$$

$$\frac{d^2 v_y}{dt^2} + \frac{q^2 B_z^2}{m^2} v_y = 0$$

$$\frac{d^2 v_z}{dt^2} = 0$$

$$\boxed{\begin{aligned} \frac{d^2 v_x}{dt^2} + w_c^2 v_x = 0 \\ \frac{d^2 v_y}{dt^2} + w_c^2 v_y = 0 \end{aligned}}$$

$$w_c = -\frac{qB}{m}$$

The static magnetic field cannot change the kinetic energy of the particle. Spatially varying magnetic field if you think of maybe even that will not be able to change the kinetic energy of the particle. I mean that statement will still have the same relevance. If you change the magnetic field to be from static to spatially varying. So, we can consider a case in which magnetic field lengths are straight and parallel.

So, we can consider something like this. We can consider a magnetic field which is like this. The field lengths are straight and parallel. Straight means there is no curvature in magnetic field. We have ruled out curvature, straight and parallel.

Parallel means there is no gradient in the magnetic field. So, we have ruled out both of these two cases. So, generally what happens is if you take a bar magnet, the magnetic field lines will converge and diverge at the poles creating something like this. We know this very well. You see here the magnetic field lengths are curved and they are parallel only at a very small distance which is also imaginary and over a very small distance but otherwise they are always curved and they are non-parallel.

So, these two are the basic characteristics of a magnetic field but we are ruling out both of them and saying that let us consider a magnetic field in which the lines of force are straight in nature and also at the same time they are parallel in nature. Now let us say the velocity of the particle is now decomposed into V perpendicular plus V parallel. What is V perpendicular? V perpendicular is the component of velocity which is perpendicular to the magnetic field. Let us say if the magnetic field is along Bz, Vx and Vy will constitute the perpendicular component of velocity and Vz which is parallel to B is V parallel. So, let us say we call this as equation number 1.

What are we doing by the way? In the earlier slide, if you take a magnetic field, we realize that if there is no electric field, the rate of change of kinetic energy will be 0. The magnetic field will not produce any acceleration or change of energy. Now we are going to see a mathematical treatment which will lead us towards the trajectory of the particle. Let us say we start with decomposing the velocity of the particle into V perpendicular and V parallel. We will write the modified expression now dV parallel by dt plus dV perpendicular by dt is equals to Q by m times V parallel plus V perpendicular cross B.

Now you have both these taking a curl on the B. We can say that V parallel cross B is 0. Can we say that? Think about it because V parallel is along the magnetic field itself. The angle is 0 and cross product is sin theta, sin 0 is 0. We are left with dV parallel by dt plus dV perpendicular by dt is equals to Q by m times V perpendicular cross B.

Now let us say we split this equation into two equations dV parallel by dt is equals to 0 which means V parallel will remain a constant and dV perpendicular by dt is equals to Q by m V perpendicular cross B. The magnetic field has no effect on the component of velocity which is parallel to it and it only affects the perpendicular component V perpendicular. So, to find out about the nature of motion under the influence of this type of a magnetic field, let us consider the magnetic field B. The configuration is Bz along z cap which means we have 0 0 Bz like this. If we have such a configuration, if you simply try to write F is before we will go to the parallel and perpendicular component just in a moment is Q times V cross B.

So, V is Vx Vy Vz B is 0 0 Bz. V cross B is ijk Vx Vy Vz 0 0 Bz. Now take this product from this vector product is I will get m dVx by dt is equals to in the direction of i cap. So, it will be Vy Bz Vy Bz Q this one and m dVy by dt is in terms of j. It will be minus Vx Bz Q and m dVz by dt will be 0. Let us say we call this set of equations as 1 is 2 is 3 is 4 and this is 5.

The component of velocity which is parallel to B is denoted with V parallel. Now because the cross product Q times V cross B acts perpendicular to z. Q times V cross B will act perpendicular to z because B is along z direction. So, it is obvious that it will be perpendicular to the z direction. So, to examine the variation of Vx and Vy with respect to time you see these two equations.

Now we have to work around these two equations. One thing is in the case of uniform static electric field we have realized things will happen only in the direction of electric field but what we have realized here is that along the direction of magnetic field the velocity will remain a constant but it will not remain a constant along the other directions which are perpendicular to z direction. So, it is like this you have x y and z the particle velocity will remain a constant along this direction but it will not remain a constant along x and y direction how will it change? It will change by these factors m dVx by dt is equal to Vy Bz Q minus Vx Bz Q. Now let us try to understand this mathematically. Now what I will do is I will take a derivative with respect to time on equation 5. So, m d square Vx by dt square is QB Vx by dt and m d square Vy by dt square I have taken a time derivative on equation on the last equation is minus QB dVy by dt.

You see you have Vy which will appear here Vx which will appear here and m d square Vz by dt square is still 0. So, let us say we call this set of equations as 6 now using 5 in 6 how are we going to use it? We have from the last equation dVy by dt. So, from this we can write we will write it here dVy by dt is minus Q by m Vx Bz and dVx by dt is Q by m Vy Bz it is coupled the x component of the velocity has velocity Vy and the y component of the velocity has the velocity Vx. Using that into these equations we will be able to write m d square Vx by dt square is equal to QB times minus QB by m Vx. We can continue writing the subscript z to indicate the direction of the magnetic field and so d square Vx by dt square is minus Q square by m square Bz square Vx or we have d square Vx by dt square plus Q square Bz square by m square Vx is equal to 0.

This is from 6a, b and c and if we do the same m d square Vy by dt square is minus QB times QB by m Vy this can also be written as m d square Vy by dt square plus Q square Bz square by m square Vy is equal to 0. We have these two very important equations. Let us take them out and write them separately. What we have is d square Vx by dt square plus Q square V square by m square Vx is equal to 0, d square Vy by dt square plus Q square by m square Vy is equal to 0. This is a very familiar equation and the third one is d square Vz by dt square will still remain to be 0.

Now, in simple harmonic oscillator or in simple harmonic motion we have seen this type of equation. So, this term that appears here which is multiplying Vx should be in the units of frequency square, d square V by dt square plus omega c square Vx is equal to 0, d square Vy by dt square plus omega c square Vy is equal to 0. Now, it is very important for us to understand this omega c. As for this substitution omega c is Q B by m. Let us say we make the sign explicit on the right hand side.

So, we say that omega c is minus Q B by m. What is this? This is angular frequency no doubt and it is conveying a powerful message actually. So, we will stop it here. We will try to understand more implications of these equations and we will try to understand how the particles trajectory will look like when it is experiencing a uniform static magnetic field. We will continue this in the next lecture where we will try to understand the implications of these equations and describe the motion of a charged particle in the presence of a uniform static magnetic field.