Plasma Physics and Applications

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Week – 03

Lecture 11: Numerical Problems on Debye Shielding -II

Hello dear students. In today's lecture, we will try to solve some numerical problems based on device shielding and plasma frequency etc. So, these numerical problems will help us understand the concept better and also increase your problem solving ability. So, as we know plasma is considered as a neutral entity. We refer to it as the quasi neutrality of plasma where within a very small microscopic regions there can be some disparity between the number of electrons and ions. But as a whole the entire plasma has equal number of electrons and ions.

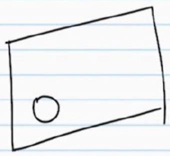
So, this idea is called as a quasi neutrality. So, electrically neutral on the whole, but the neutrality may not be maintained over very small microscopic regions. So, there is a chance that at some point there are more number of electrons or ions. So, within this plasma particles the electrons and ions move around because of their thermal energies because of the temperature that is there.

So, these particles can be moving around from one point to another point. And they may produce local concentration of charges. We know this and when there is a local charge concentration that means an excess amount of electrons for example, immediately an electric field would be set up and this electric field would make the electrons oscillate because of their thermal energy and thereby the neutrality is established. At the same time it also gives you the provision to understand the plasma frequency. Most importantly something that we have to understand very clearly is that the plasma generally allows charge separation only to a certain distance where there can be a balance between the thermal energy of the particle.

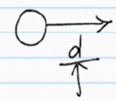
Ex-1 $\quad n_e = n_i = 10^{18}/m^3$ $\qquad$ Plasma

→ Small spherical volume with $r = 0.01\,m$

— ion density is larger than e^- density by 1%



1) KE & Electrostatic PE



2) d should be microscopic

Sol number of ions

$$n_i = 1.01 n_e$$

$$\rho = (n_e - n_i)q. = (n_i - n_e) \times q$$

$$n_i > n_e$$

$$q = \frac{4}{3}\pi r^3 \times (n_i - n_e)\, q$$

$$\frac{q}{V} = \rho \Rightarrow q = \rho \times V$$

Let us say the kinetic energy because of the temperature of the particle and the electrostatic potential energy. So, these two energies when they are matched so that means if you are separating electrons over a particular distance d. So, plasma can allow charge separation only to a certain value of d where the kinetic energy or the thermal energy of these electrons is in correspondence or is approximately equal to the electrostatic potential energy. If it is very large distance then the electrostatic potential energy is there but the kinetic energy of the particles may not be sufficient to cover this or to nullify this charge separation. So, this is a limiting condition for setting up of plasma frequency or for that matter the charge deviation from charge neutrality.

$$q = \frac{4}{3}\pi r^3 \times (n_i - n_e) \times q$$

$$q = \frac{4}{3}\pi (0.01)^3 \times (1.01 - 1) n_e \times q$$

$$q = \frac{4}{3} \times 3.14 \times (0.01)^3 \times (0.01) \times 1 \times 10^{+18} \times 1.6 \times 10^{-19}$$

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$\phi = \frac{9 \times 10^9}{0.01} \times \frac{4}{3} \times 3.14 \times (0.01)^3 \times (0.01) \times 1 \times 10^{18} \times 1.6 \times 10^{-19}$$

$$\boxed{\phi = 6000\,V}$$

So, we can say that plasma can allow charge separation only to certain distances where or up to which there can be a balance between the thermal energy of the particles and the electrostatic potential energy which has resulted by the charge separation. And second most important thing is as a result plasma cannot maintain deviation from the charge neutrality over macroscopic distances but it can only maintain or it can only allow deviations from the local charge neutrality over a very small microscopic distances. So, this distance of separation or over which the charges are being pulled apart should be microscopic in nature. So, if you extend it beyond a distance this deviation from charge neutrality may not be valid. So, in order to understand this concept better so we mean saying that whenever you displace electrons immediately electric field would be set up and the electron will be asked to or electron will be coming back against the direction of electric field to nullify this charge separation.

Q-2 For a typical fusion reactor, the electron density is $n_e = 10^{21}\ m^{-3}$ and temperature 10 keV. Verify if this can be called as plasma.

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{n e^2}}$$

1 eV = 11600 K

$$= \sqrt{\frac{8.854\times10^{-12}\times 10^4 \times 1.6\times10^{-19}}{10^{21}\times(1.6\times10^{-19})^2}}\ m$$

1) $\omega_c \tau_p \gg 1$

2) $N_D \gg 1$

3) $L \gg \lambda_D$

$$\lambda_D = 2.3\times10^{-5}\ m$$

$$N_D = \frac{4}{3}\pi\lambda_D^3 \times n$$

$$= \frac{4}{3}\times 3.14 \times (2.3\times10^{-5}) \times 10^{21}$$

$$\boxed{N_D = 5.1\times10^{7}}$$ ✓ Plasma

So, in order to understand this concept or the setting up of electric field or how large are these electric fields going to be we can take a simple example let us say we consider. So, this is example number 1 for today's class. So, let us consider a uniform plasma with number of electrons equal to number of ions which is of the order of 10 to the power of 18 per meter cube and this is the plasma description. And let us consider a small spherical volume with radius R is equals to 0.

01 meter. So, let us say we have a small spherical volume inside this plasma with radius R is equals to 0.01 meters and within this volume we have number of ion that is the ion density is larger than the electron density how much by 1 percent. Then we have to find out what will be the magnitude of the electric field that will be produced within this sphere. So, we have the plasma and inside the plasma we have considered a small region whose spherical region whose radius is something like 0.01 meter and with only within this spherical region ion density is larger than electron density by 1 percent.

Q-3

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{r}\cdot e^{-r/\lambda_D} \Longrightarrow \vec{E} \qquad \vec{E} = -\vec{\nabla}\phi$$

$$\frac{d}{dr}\left[\frac{1}{r}e^{-r/\lambda_D}\right]$$

Find $\vec{E}$

Sol

$$\vec{E} = -\hat{r}\frac{d\phi}{dr} = \frac{q}{4\pi\epsilon_0}\,\hat{r}\left[-\frac{1}{r^2}e^{-r/\lambda_D} - \frac{1}{r}\cdot\frac{1}{\lambda_D}e^{-r/\lambda_D}\right]$$

$$\hat{r} = \frac{\vec{r}}{|r|}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}+\frac{1}{\lambda_D}\right]e^{-r/\lambda_D}\cdot\frac{\vec{r}}{r^2}}$$

$\Downarrow$

$\rho(r)$

$\rho(r),\ \phi,\ \vec{E}$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

That means if there are 100 electrons in that region the number of ions would be just 1 more that is not 1 electrons will be there in terms of percentages. So, now you have to think about the total number of electrons and then we have to find out how much amount of electric field will be produced by this charge separation. So, there are less number of electrons in comparison to the ions. So, number of ions in this case the solution the number of ions n i is equal to 1.

01 n e. How much is it? 101 percent of electrons 101 divided by 100 into n e. So, the total charge density let us say rho is n e minus n i times the charge this is the formula. But in our case it is number of ions is greater than number of electrons. So, the total charge is it has to be total charge has to be written as n i minus n e number of particles per unit volume multiplied by particle charge individual particle charge. So, we will have charge per unit volume.

$$\rho(r) = \epsilon_0 (\vec{\nabla} \cdot \vec{E})$$

$$= \frac{q}{4\pi} \vec{\nabla} \cdot \left[\frac{\vec{r}}{r^3} e^{-r/\lambda_D} + \frac{\vec{r}}{\lambda_D r^2} e^{-r/\lambda_D} \right]$$

$$\vec{\nabla} \cdot \phi \vec{A} = \vec{\nabla}\phi \cdot \vec{A} + \phi \vec{\nabla} \cdot \vec{A}$$

$$\rho(r) = \frac{q}{4\pi} \left[\vec{\nabla}(e^{-r/\lambda_D}) \cdot \frac{\vec{r}}{r^3} + e^{-r/\lambda_D} \vec{\nabla} \cdot \frac{\vec{r}}{r^3} + \frac{1}{\lambda_D} \vec{\nabla}(e^{-r/\lambda_D}) \cdot \frac{\vec{r}}{r^2} + \frac{1}{\lambda_D} e^{-r/\lambda_D} \vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right]$$

$$\vec{\nabla}(e^{-r/\lambda_D}) = -\frac{1}{\lambda_D} e^{-r/\lambda_D} \hat{r}$$

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = -\vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{r}\right)$$

$$= \nabla^2 \frac{1}{r}$$

$$= 4\pi \delta(r)$$

So, if you want to find out the charge then we have to see the charge can be 4 by 3 pi r cube into volume into the charge density that will be n i minus n e times q charge per unit volume is charge density charge is equals to charge density multiplied by volume. Since it is a spherical volume the volume becomes 4 by 3 pi r cube. Let us take it to the next slide I will rewrite the expression q is equals to 4 by 3 pi r cube into n i minus n e multiplied by the charge. So, q is equals to 4 by 3 the radius of the sphere is 0.01 meter cube n i is 1.

01 times n e 1.01 minus 1 times n e times q. So, q becomes 4 by 3 into 3.14 into 0.01 cube into 0.01 into 1 into 10 to the power of plus 18 charge is 1.

6 10 to the power of minus 19. This is the charge what is this? This is the additional charge. Now, the potential that will be created out of this has to be phi is equals to 1 by 4 pi epsilon naught q by r. So, potential phi the value of 1 by 4 pi epsilon naught is 9 10 to the power of 9 into q, q is all of this divided by r is 0.

01 meters. So, that will be 4 by 3 into I am just rewriting it all of it 0.01 cube into 0.01 into 1 into 10 to the power of 18 into 1.6 into 10 to the power of minus 19. So, if you use a calculator what you will realize is that this will be this entire product on the right hand side will be approximately equal to 6000 volts.

$$\rho(r) = \left[-\frac{1}{\lambda_D} e^{-r/\lambda_D} \cdot \frac{1}{r^2} + e^{-r/\lambda_D} \vec{\nabla} 4\pi\delta(r) - \frac{1}{\lambda_D^2} e^{-r/\lambda_D} \frac{1}{r} + \frac{1}{\lambda_D} e^{-r/\lambda_D} \cdot \frac{1}{r^2} \right]$$

$$\boxed{\rho(r) = q\left[\delta(r) - \frac{1}{4\pi\lambda_D^2 r}\right] e^{-r/\lambda_D}}$$

The potential is 6000 volts. Now, the point that I am trying to convey is that the imbalance between the number of electrons and ions within the sphere is only 1 percent. The number of ions are larger only by 1 percent that means, electrons are displaced how many electrons only 1 electron per 100 ions is displaced from this region and they have been kept away only by a very small distance actually. And because of that the amount of potential that develops is very high you see this 6000 volts is a very large value of potential. That means that like we said before whenever you try to displace electrons immediately large electric fields would be set up which will try to nullify the charge moment immediately.

So, this is how we can understand the setting up of electric field by the displacement of electrons and how plasma oscillations a natural consequence of this are relevant to maintain charge neutrality inside the plasma. So, plasma is neutral in a macroscopic sense scale, but neutrality may not be maintained over very small microscopic lengths and as well as time scales. Now, let us take one more example to understand concept of device length or how the plasma criteria can be used to maintain or to qualify a new an ionized gas as plasma. Example number 2, let us say for a typical fusion reactor the electron density is n e is equal to 10 to the power of 21 per meter cube and temperature is 10 kilo electron volts. Verify if this can be called as plasma.

So, for a typical fusion reactor the electron densities are of the order of 10 to the power of 21 per meter cube and the temperature is 10 keV. Now we know that we use the scale of electron volts for measuring or for quantifying the temperatures of plasma where the equality is 1 electron volt is equal to 11600 temperature units that is in Kelvin. Now, this is an ionized gas no doubt, but should we call it as plasma or not? That is the question. Now, for it to be called as plasma we require three conditions. One omega c tau p should be much greater than 1.

Number 2 is the plasma parameter, the number of plasma particles or ionized particles within the Debye sphere must be very large and the length scale of the plasma should be very large in comparison to the Debye's length. So, let us start from the Debye's length lambda d is equal to square root of epsilon naught k B T by n e square. We know the meaning of each of the variable or constant that appears on the right hand side. So, we can directly substitute.

So, epsilon naught is 8.854 10 to the power of minus 12. k B T is the energy. So, temperature is used in electron volts. So, the Boltzmann's constant which is should also be converted into electron volts. So, we can write it as 10 to the power of 4 is the temperature and if you write it in electron volts, you have to write it 1.

6 10 to the power of minus 19. One electron volt is equal to 1.6 10 to the power of minus 19 joules. n is 10 to the power of 21, charge density is 10 to the power of 21, electron charge is 1.6 10 to the power of minus 19 e square.

The units are meters. So, this will become 2.3 10 to the power of minus 5 meters. So, lambda d is of the order of this. What it means is that if you try to put some external electric field, this is the distance up to which the electric field can penetrate. This electric field will be invisible.

But this does not guarantee the condition of plasma. So, we have to see number of particles inside the plasma is 4 by 3 pi lambda d cube into n, which will be 4 by 3 into 3.14 into lambda d cube that is 2.3 10 to the power of minus 5 into 10 to the power of 21.

So, if you do this, you will get 5.1 10 to the power of 7. What is this? Nd, Nd is called as the plasma parameter. So, you can see that the plasma parameter is very large, which means within the Debye sphere, there are a lot of charged particles, electrons or ions. So, conclusively we can say that since this condition is valid, we can say that this particular ionized gas which exists within a reactor can be given the name of plasma.

So, this is the basic idea. So, similarly we can using the formulae of lambda d or simplified formulas of Debye s length, plasma frequency etcetera, we can quantify whether a particular gas would be a plasma or not. One more very interesting problem that we can take is basically kind of a derivation which can also be considered as an extension to the Debye s length derivation itself. So, we know that the question or question number 3, we know that the plasma potential phi or the Debye s potential phi of r is given as 1 by 4 pi epsilon naught q by r into e to the power of minus r by lambda d, where lambda d is the Debye s length. If this is the potential, find out or derive an expression for the electric field. So, we know that electric field is the negative potential gradient.

So, e is equals to minus del phi. We have to find out the electric field now. But since the coordinate system is in terms of r, we have to use an appropriate formula. So, we can write the question is find the electric field, the solution. So, we know that the electric field E is minus r cap d phi by dr.

Why? Because of this. So, we can write it as using all of this potential inside the bracket. So, we can bring out q by 4 pi epsilon naught r cap times when we apply differentiation on 1 by r into e to the power of minus r by lambda d. So, it is d by dr of 1 by r e to the power of minus r by lambda d. The direction is already given r cap. So, this is u v d by dr of u v u d v plus v d u.

So, that is what I am going to write here minus 1 by r square e to the power of minus r by lambda d minus 1 by r into 1 by lambda d e to the power of minus r by lambda d. So, using the fact that r cap is r by mod r, we can write the electric field as q by 4 pi epsilon naught 1 by r plus 1 by lambda d e to the power of minus r by lambda d multiplied by r cap by r square. So, one out of this term and this term have taken 1 by r common it came out and this r cap is r by mod r which is again r that is why we have r square outside and the remaining terms inside this. So, this is the electric field. Now, we can simplify it further and we can also find out what is the charge density.

So, this is the electric field which is created within the device potential. So, we can also go ahead and find out what is the charge density rho of r. Let us take rho of r. So, how do we calculate rho of r? We know phi, we know electric field, the potential and the charge density. So, knowing potential we have calculated the electric field.

How can we calculate the charge density? Now having known the potential the electric field, we can now calculate the charge density by using the formula del dot E is equals to rho by epsilon naught which is the first Maxwell equation. So, what I will do is rho of r the charge density is epsilon naught times del dot E within the electric field that we have just derived it will be q by 4 pi del dot r by r cube e to the power of minus r by lambda d plus r by lambda d r square e to the power of minus r by lambda d. Now, we have to calculate the divergence of this all this quantity which appears in the bracket. So, we will use a standard vector identity which is del dot phi a. If we have a situation where we have to calculate the divergence of a vector being multiplied by a scalar function, then the formula is del phi dot a plus phi times del dot a.

First one is find the gradient then take a divergence with the vector then put the scalar quantity outside then again you will get a scalar. So, eventually the entire summation is a scalar. So, the charge density rho of r is q by 4 pi I will just write the entire algebraic expansion you can do it without looking at the board e to the power of minus r by lambda d times r cap by r cube plus e to the power of minus r by lambda d. We will have four terms because there are two terms within the divergence bracket 1 by lambda d del of e to the power of minus r by lambda d r cap by r square plus 1 by lambda d. Now, using the fact that del of e to the power of minus r by lambda d is nothing but minus 1 by lambda d e to the power of minus r by lambda d and the direction would be along r cap and del dot r by r cube can be rewritten as minus del dot del of 1 by r, del of 1 by r is minus 1 by r cube we are multiplying with r.

So, the vector will stay outside and then you will have which will be equal to the same thing. So, which can be written as del square of 1 by r. We know that del square of 1 by r is 4 pi times delta for the direct delta function. Using this into this we can write the charge density rho of r is equals to minus 1 by lambda d e to the power of minus r by lambda d 1 by r square plus e to the power of minus r by lambda d. We will have cancelling the relevant term we will have Q times delta r minus 1 by 4 pi lambda d square r e to the power of minus r by lambda d.

This is the charge density. So, what we have done is we have given the device potential we are able to find out the electric field and the charge density which is actually responsible for the shielding effect. Thank you. .