Plasma Physics and Applications

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Week – 02

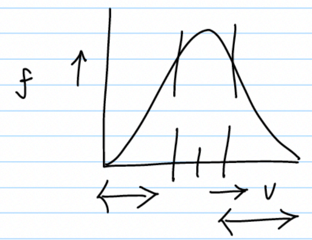
Lecture 10: Plasma as a Gas & Distribution of Velocities

Hello dear students. In this lecture, we will try to understand how the energies are distributed in a plasma considering the fact that plasma can be treated as a gas. So, in the case of a gas which is in thermal equilibrium, let us say we are going to discuss about plasma as a gas. So, when you say gas is in thermal equilibrium that implies that the particles can have all possible or all probable velocities. And how these velocities are distributed can be given generally by a distribution function And for an ideal gas, we can take the Maxwell or Maxwellian distribution. Let us try to understand what is a distribution.

What is a distribution? Let us say if the particles there are 10 to the power of 20 atoms or molecules are there inside this gas chamber, the system that you have taken, then not all particles will have the same velocity, it is the highest velocity, not all particles will have the lowest velocity. They will obey a distribution function. So, then let us say it will look something like this, which means most of the particles if you have this velocity and if you have this function, which is on the or let us say number of particles on the y axis. What it says is that if you there will be lot of particles will be there which will have an average velocity, the smallest velocities will be possessed by some molecules, some particles and the largest velocities will also be possessed by some.

Plasma as a gas

Maxwellian Distribution



$$f(u) = A \exp\left(\frac{-\frac{1}{2}mu^2}{k_B T}\right)$$

$f(u)$: number of particles with velocity u & $u+du$.

$A\, e^{-E/k_B T}$

'n'

$$n = \int_{-\infty}^{\infty} f(u)\,du$$

Now is it possible that a similar type of distribution is also expected in plasma because plasma is also an ionized gas. For this, we have to consider if all the particles are moving in the same direction or not. Of course, you can make all the particles to move in the same direction by applying an external magnetic field that is not a very difficult thing. If you take one dimensional picture for a plasma, the maxwellian distribution function can be written as, let us say F of u is A exponential minus half m u square by k B T. This is nothing but A e to the power of minus e by k B T that we are familiar with.

What does it tell? It tells that for given velocity, what is the probability? Let us say for larger velocities, the distribution function will be smaller, the number of particles which will have this velocity will be smaller and things like that. Where k B is the Boltzmann's constant and F u the function can be the number of particles per unit volume with a velocity between u and u plus du. What is F u? F of u is number of particles with velocity between u and u plus du. What is u? u is by the way velocity in one dimension. If you want to find out the density of number of particles per unit volume, let us say if you are interested to find out what is n, the number of particles per unit volume, then what you have to do is you take n is equal to minus infinity to infinity F u du.

For a normalized 1-D

$$\int_{-\infty}^{\infty} f(u)\,du = 1$$

$$\int_{-\infty}^{\infty} A\, e^{-\frac{mu^2}{2}/k_BT}\, du = 1$$

$$A = \sqrt{\frac{m}{2\pi kT}}$$

Width of distribution function is a measure of 'T'

$T \propto$ Average KE of system

You integrate the entire function to find out the total number of particles per unit volume. Now for normalized one dimensional Maxwellian distribution function, what you can write is for the normalized one dimensional Maxwellian distribution function, you can write it as minus infinity to infinity F u du will be equal to 1. When you say it is equal to 1, you are referring to normalization. In a sense, you are writing minus infinity to infinity A e to the power of minus m u square by 2 divided by k B T du is equal to 1. If you solve this integral, you will realize the value of constant is square root of m by 2 pi k T.

It is in the form of e to the power of minus x square multiplied by a constant. You will realize that the value of A is something like this. Now how is this distribution function important? For us the width of this distribution function is a measure of temperature. What is temperature by the way? Temperature is a measure of the mean molecular energy of a sample. If you take a gas, the mean molecular energy is represented in the units or in the language of temperature.

$$E_{av} = \frac{\int_{-\infty}^{\infty} \frac{1}{2} m u^2 f(u)\, du}{\int_{-\infty}^{\infty} f(u)\, du} \leftarrow$$

$$\frac{1}{2} m v_{th}^2 = k_B T \leftarrow \underline{(a)}$$

$$\Rightarrow du = v_{th}\, dy.$$

$$v_{th} = \sqrt{\frac{2 k_B T}{m}} \qquad y = \frac{u}{v_{th}} \Rightarrow dy = \frac{du}{v_{th}}$$

$$f(u) = A \exp\left(\frac{-u^2}{v_{th}^2}\right)$$

Temperature is proportional to the average kinetic energy of the system because both of them convey the same message about the gas. Temperature is proportional to average kinetic energy of the system. It is proportional. We are not saying that the average kinetic energy is temperature. These two things are directly proportional to each other.

So, the average kinetic energy is a measure of the temperature. How do we calculate the average kinetic energy? Let us say E average is integral minus infinity to infinity half m u square f u du divided by minus infinity to infinity f u du. Now let us say the particle has some thermal velocity and because of it some kinetic energy m v t h square is equals to k B T because by the virtue of its temperature, it has this energy k B T and this energy must be equal to the kinetic energy of the particle. So, we can write the thermal velocity that means the particles motion because of its temperature can be represented in terms of the thermal velocity. The thermal velocity is 2 k B T by m.

$$E_{av} = \frac{\int_{-\infty}^{\infty} \frac{1}{2} m y^2 v_{th}^2 A \exp(-y^2)\, v_{th}\, dy}{A \int_{-\infty}^{\infty} \exp(-y^2)\, v_{th}\, dy} \quad \text{--- (1)}$$

$$E_{av} = \frac{\frac{1}{2} m v_{th}^3 A \int_{-\infty}^{\infty} \exp(-y^2) \cdot y^2\, dy}{A\, v_{th} \int_{-\infty}^{\infty} \exp(-y^2)\, dy}$$

$$\int_{-\infty}^{\infty} y \cdot \left[\exp(-y^2)\right] y\, dy$$

Now let us say for simplicity, we are trying to solve this E average. Let us say we write y ratio as u by v t h. u is the one dimensional velocity, v t h is the thermal velocity. So, using this representation of y is equals to u by v t h, we can write the distribution function f u as A times exponential minus u square by v t h square because you have written k B T the thermal energy in the units of half v t h square. So, E average using this we can write E average dy is equals to du by v t h which implies du can be replaced as v t h dy.

Using that we can write the average energy of the sample as minus infinity to infinity half m y square v t h square. Where did we get this? Because u is equals to y times v t h from this, making an appropriate substitution m y square v t h square A exponential of minus y square v t h dy. So, this is du and this is after substitution and this is again substituting y is equals to u by v t h. The same expression that you see here is now written like this using the substituted variables. Once we know what it is, we can get it of all these things divided by A times integral minus infinity to infinity exponential of minus y square v t h dy.

$$= \left[-\frac{1}{2}\left[\exp(-y^2)\right] y \right]_{-\infty}^{\infty} - \left[\int_{-\infty}^{\infty} -\frac{1}{2}\exp(-y^2)\,dy \right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\exp(-y^2)\,dy. \qquad (2)$$

Using 2 in 1

$$E_{av} = \frac{\frac{1}{2}\cancel{\int_{-\infty}^{\infty}\exp(-y^2)\,dy} \times m A v_{th}^3 \times \frac{1}{2}}{A v_{th}\cancel{\int_{-\infty}^{\infty}\exp(-y^2)\,dy}}$$

We try to evaluate this integral E average is half m v t h cube A. The integration is with respect to y. So, it only makes sense we have all the terms which involve y inside the integral and everything else a constant be outside the integral exponential of minus y square times y square dy divided by A v t h. If you are trying hard to follow what I am doing on the board, you can try working it out all the steps I have not missed any algebra steps in between. So, you can work out by just looking at what I am trying to do.

So, this is what we have to solve now. Let us see this integral of minus infinity to infinity y times exponential minus y square y dy. This integral can be evaluated like this which will be equal to minus half exponential of minus y square times y between minus infinity to infinity minus integral u dv minus half exponential minus y square dy. So, this will simplify to half minus infinity to infinity exponential minus y square dy using this back into the original equation which is let us say we call this as 1 using this 2, using 2 n 1. We can write the E average is half integral minus infinity to infinity exponential minus y square dy multiplied by m A v t h cube times 1 by 2 divided by A v t h.

$$E_{av} = \frac{\frac{1}{2} m A v_{th}^3 \times \frac{1}{2}}{A v_{th}}$$

$$E_{av} = \frac{1}{4} m v_{th}^2$$

Using (a)

$$\frac{1}{4} m v_{th}^2 = \frac{1}{2} k_B T$$

$$\underline{\underline{\text{Average } K.E = \frac{1}{2} k_B T}}$$

Now, we can cancel these two things and as a result what we are left with is half m A v t h cube divided by A v t h. Writing the remaining terms E average is equals to half m A v t h cube into half divided by A v t h or 1 by 4 m v t h square. What is this? This is E average. The average kinetic energy is 1 by 4 m v t h square. Is there any relation between the kinetic energy and the thermal energy earlier that we have seen? Half m v t h square is k B T using let us say we name this expression as A using A.

We can write 1 by 4 m v t h square is equal to half k B T because half m v t h square was k B T, half of that is half k B T. That is we can say that the average kinetic energy along one dimension is half k B T. Now, we can extend this into three dimensional picture or we take the distribution function. It used to be f of u. Now, we have f of u v w is equals to A times exponential minus half m times u square plus v square plus w square divided by k B T and you evaluate the value of constant as n times m by 2 pi k B T raise it to the power of three half.

If you follow the same process as we have followed the average energy E average is triple integral minus infinity to infinity. This one f times the exponential part times the energy all that divided by integral over minus infinity to infinity A the exponential part and d u d v d w same d u d v d w. If you do all that then the average kinetic energy along three dimensions would be 3 by 2 k B T. So, E average is 3 by 2 k B T average energy along three dimension as we can interpret that average energy is half k B T along one degree of freedom or when velocity is one in one direction when the velocity is in all the directions we can say that it is 3 by 2 k B T. So, this is just to know that how the energy is distributed in plasma or in a gas.

$$f(u,v,w) = A \exp\left[-\frac{1}{2}m\left(u^2+v^2+w^2\right)/k_BT\right]$$

$$A = n\left(\frac{m}{2\pi k_BT}\right)^{3/2}$$

$$E_{av} = \frac{\iiint_{-\infty}^{\infty} f \exp \cdot E \; du\,dv\,dw}{\iiint_{-\infty}^{\infty} A \exp \; du\,dv\,dw}$$

$$\boxed{E_{av} = \frac{3}{2}k_BT}$$

Now, plasma is considered as a gas. So, in plasma physics we have a practice of writing temperature not in the unit of Kelvin but rather in the units of temperature in the units of electron volt. So, E average is 3 by 2 k B T a gas at 1 Kelvin will have an energy of k B T. So, this is the energy. You see this is the energy how much energy is it? So, energy has to be let us say if it is 1 if energy is 1 eV what it means is 1.

6 10 to the power of minus 19 joules. k B T is this. Now, the k B T the thermal energy if it is equal to 1 electron volt which means 1.6 10 to the power of minus 19 joules. Now, the temperature that is responsible for this energy can be calculated by dividing 1.

6 into 10 to the power of minus 19 divided by 1.38 into 10 to the power of minus 23. I have substituted the value of what is it the Boltzmann constant. Now, if you do this you will realize that 1 electron volt is equal to this much how much 11600 Kelvin. So, this is the conversion factor.

So, both of these two things are same 11600 Kelvin how much is it equivalent in the units of energy it is equal to 1 eV or you can use the conversion factor wherever it is possible. So, because the scale of temperatures in plasma physics is very high. So, you often encounter very large temperatures. So, it is convenient to express them in the units of electron volts rather than in the units of Kelvin. So, we can keep something in mind like this 1 electron volt plasma is equal or 1 electron volt plasma is something that is existing at a temperature of 11600 Kelvin.

So, temperature that is required to produce plasma is very high. We already know how to produce plasma solid to liquid to gas to plasma. So, that means that the amount of energy that you have to spend to create plasma is very high and this very high values of temperature are often represented in the units of electron volt, but not in the units of Kelvin. So, for example, many plasmas can have temperatures of let us say 10, 100, 1000 Kelvin. So, rather than writing this like this we write it is equivalent which is nothing but 100 electron volt plasma.

T in Units of eV

$$\underline{\underline{1\,k}} = \underline{\underline{k_B T}} = \underline{1\,eV} = 1.6 \times 10^{-19}\,J$$

E

$$k_B T = 1\,eV = 1.6 \times 10^{-19}\,J$$

$$T = \quad = \frac{1.6 \times 10^{-19}}{1.38 \times 10^{-23}}$$

$$\boxed{1\,eV = 11{,}600\,k}$$

$$\underline{10{,}00{,}000}\,k = \underline{100\,eV}$$

So, the notation that we use in plasma physics is that it is a plasma at 100 electron volt. What it means is that 1 electron volt is equal to 11600 Kelvin you can make the conversion and say that the plasmas temperature is so much. So, in all the problems when we discuss in plasma physics, we always mention the condition of plasma in the units of electron volts and number densities per unit volume. So, this is some information about plasma as a gas and how energies are distributed in plasma. What we have learnt is average kinetic energy along 1 degree of freedom is half kBT and the second most important fact that we have learned is in plasma physics the temperatures are represented in the units of electron volts, but not in the units of Kelvin.