Newtonian Mechanics With Examples

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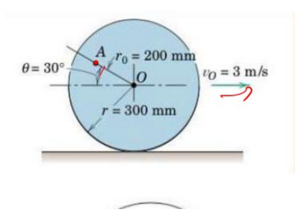
Lecture-44

In the last lecture, we considered an example of the rolling motion of a wheel. In that case, we analyzed the problem with respect to the center of the wheel. Now today, I am going to show that it is not necessarily that you have to always analyze the problem with respect to the center of the wheel. The reference point about which we are going to describe the rotation of the object can be anything. So, in this example,



The wheel in the figure rolls to the right without slipping, with its center O having velocity $\mathbf{v}_O$ = 3 m/s to right. Locate the instantaneous center of zero velocity and use it to find the velocity of point A indicated in the figure.

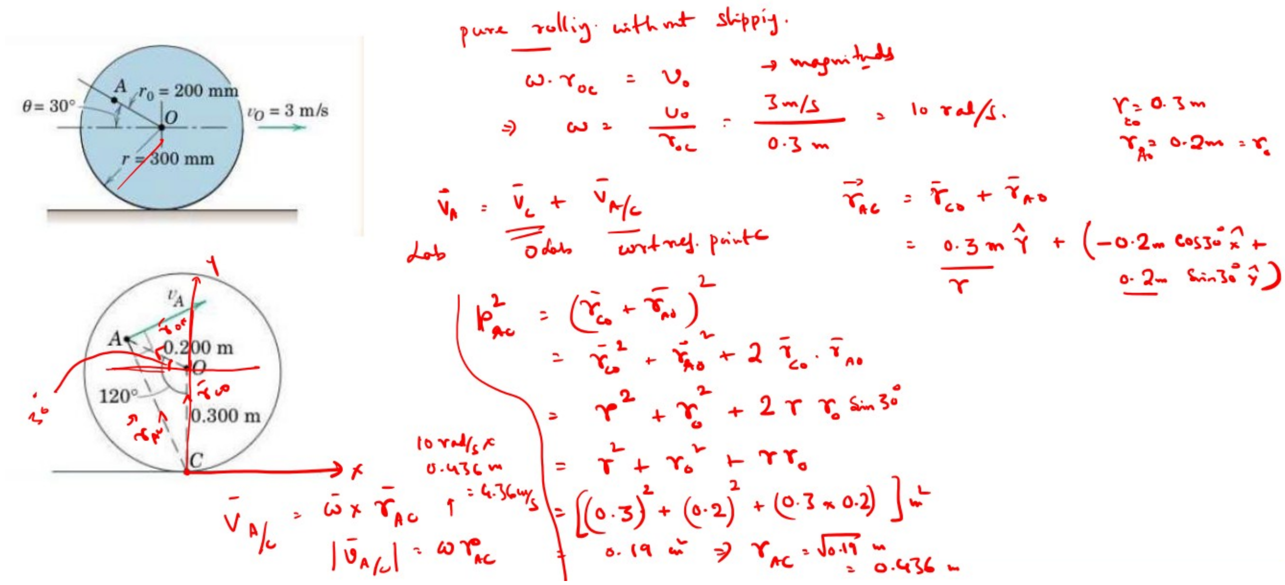
Ans: (a) $\boldsymbol{\omega} = \omega\,\mathbf{k} = -10$ rad / s (b) $\mathbf{v}_{A,C}$ = 4.36 m/s, perpendicular to AC

So, here is our strategy. So, the instantaneous in a pure rolling without slipping, the, this is the point where the velocity is 0.

So again, just to remind ourselves, this is going to be our system and we are analyzing this problem in the lab frame, some fixed origin frame in which this we see the wheel to roll. Now, first let us quickly estimate the angular velocity of the wheel. So, this we have done in the previous example in the last lecture. So, we are just going, in the pure rolling without slipping, we got the relation that the ωr_{OC} must be equal to v_0. These are all magnitudes.

So, this we learnt from our last, this is our last example. So, this gives the angular velocity which is same for all the points (assuming this wheel is a rigid body) is v_0 / r_{OC}.

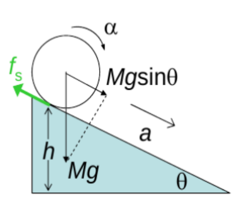
Now, we want to write this velocity of the point A with respect to the velocity about the C, so in the previous example, we decomposed it as a velocity of a reference point O+the relative velocity with respect to O. Today, we are going to do it in a different reference point. So, the velocity of the reference point C+the relative velocity of A with respect to C. Now this is 0 in the left frame. So, this is in the left frame and this is relative velocity with respect to reference point C.



Now, if we analyze this problem with respect to C, it is just a pure rotation about C. So then what is this? So the A will move in a circle with radius AC. Now how much is AC? Let us first get that. So, in this example, from this picture, so let us say if we draw a line here, from the vector, we can estimate this AC using the vector analysis. Let us call it r_{AC}.

So, then the velocity of A with respect to C is just $\omega \times r_{AC}$. And in the magnitude, if we look at the magnitude, so the direction is given here, the magnitude is easy to calculate, which is 4.36 meter per second, which is the answer.

Now let us take a very interesting, another example of rolling motion.



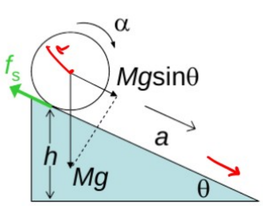
Racing Shapes: We have two objects, a solid cylinder and a solid sphere, both with the same mass, M and radius, R. If we release them from rest at the top of an incline, which object will win the race? Assume the objects roll down the ramp without slipping. Incline plane as friction.

So, in this case, we have a situation where these objects will have angular acceleration, which is given by α. Now how to analyze this problem? So, we will see that the answer depends on this ratio, the I/MR^2, where I is the moment of inertia of the object.

So, what kind of motion is it? So, in this case, the natural choice is to look at the axis of rotation, will be the axis of symmetry of the object, which is passing through the diameter in the case of sphere and the axis of cylinder, in the case of cylinder, which is going into the plane, into the screen. So, in this case, we, in our lab frame, there is some translation and also some rotation. So, the motion of the object, the cylinder or sphere is a combination of translation and rotation. So, the natural thing is to analyze it as a reference point as a center of mass of the lab frame.

So, we divide the motion into two parts. The first part is translation of the center of mass. The second part is the rotation of the center, about the center of mass. Now what determines the translation of the center of mass? So, this object is our system. So, on these, there are two forces acting.

One is the friction between the plane and the object, and the second is the gravity, which is acting downwards. Then the net force, as you know, is, so the translation of the center of mass is determined by the net force acting on the system. So the net force acting on the system is Mg, so we are taking this, this is the direction along the, down the inclined plane. In that direction, the net force that is acting is Mgsinθ - f_s. And this determine the acceleration of the center of mass.



CM frame, Lab frame

System: Sphere/cylinder

$Mg\sin\theta - f_s = M a_{cm}$ —(1)

$f_s R = I\alpha$ —(2)

$a_{cm} = \alpha R$ —(3)

✓ Translation of CM

✓ Rotation about CM

✓ Roll without slip

$$a_{cm} = g\sin\theta - \frac{f_s}{M} = g\sin\theta - \frac{I\alpha}{MR} = g\sin\theta - \frac{I}{MR}\frac{a_{cm}}{R}$$

$$\left(1+\frac{I}{MR^2}\right)a_{cm} = g\sin\theta$$

$$a_{cm} = \frac{g\sin\theta}{1+\frac{I}{MR^2}}$$

$\alpha = \frac{d\omega}{dt}$

$L = I\omega$

$\frac{dL}{dt} = I\alpha$ = Torque

Object with smallest I/MR^2 wins!

Now about the rotation, about the center of mass, so we saw, so the α=dω/dt. So previously we got this relation. In this case, this is a particularly simple case where the angular velocity is fixed and passing along the axis of the object, and the angular momentum and angular velocity are in the same direction. Because the axis of rotation is an axis of symmetry of the object, hence it is a particularly simple case. So, we have this particular relation, and in the dL/dt, which represents the torque on the object, will be given by Iα, and this represents the torque.

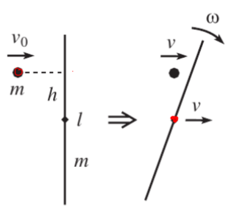
So, what is the torque on the object? So, the torque on the object is just f_sR, where R represents the radius. So, now we have three variables a, f_s and α. So, we need one more equation, and that is the condition of rolling without slipping. So, the condition of rolling without slipping is that a=αR. So, now we have three equations in three unknowns, we can solve it.

So, the acceleration depends on several parameters, the angle of inclination, strength of acceleration due to gravity, the moment of inertia of the object, the radius of the object, and the mass of the object. So there are several parameters, but they sort of combine to make a sort of this g and θ form a single group of parameter, and I, M and R^2 form as another single group of parameter. So the acceleration is essentially given by two parameter, and this shows that the object with smallest I/MR^2 wins. Now, here I mention one interesting point, one interesting variation. Here we assume that the inclined plane is a plane, so it just go down in a straight line.

What happens, but if you sort of think of something like a roller coaster game ride, then the path is not really a straight line, but or a plane, but it has some ups and downs. So interesting variation is that if you start this cylinder and sphere from a same height and same position, and but the path becomes curvy, it has ups and downs, and then if you make the race, then which object will win. So this is a interesting variation of this problem. There are several YouTube videos which sort of shows you this kind of demonstration, experimental demonstration, and so I invite you to search YouTube and look at some of those videos, and that will give you some interesting insight about this kind of problem. Now we take a new different kind of situation, and that very important class of problem we have discussed before.

So in earlier we discussed about collision problem. So now we generalize the collision problem in the situation that now we allow rotational motion as well during the collision. So the typical problem that example that I am having in mind is something like a ball hitting a bat. So this is a simplified version of this problem. So this example, this is an example of an elastic collision with rotation.

Example 41: elastic collision with rotation



A mass m travels perpendicular to a stick of mass m and length l, which is initially at rest. At what location should the mass collide elastically with the stick, so that the mass and the center of the stick move with equal speeds after the collision?

Ans. $h = \frac{\ell}{\sqrt{6}}$.

So the new thing in this analysis as we see is that now we need to also consider the conservation of angular momentum.

So we have to the effect of rotation. So let us understand what is the motion we are trying to analyze here. So the motion that we are trying to analyze is that this mass m, a ball comes and hits the stick, and after the collision the ball is keep moving, keep continue moving in the same direction with some velocity v, and the velocity v will be different from the velocity before the collision, because of the conservation of momentum. The stick which was initially at rest will now start moving. Now the motion of the stick is now a kind of, so it is doing some sort of a movement like this. So this center of mass of the stick is moving in a straight line, a translation, doing translation along a straight line as shown in the figure, and the rest of the stick is actually rotating about the center of mass.

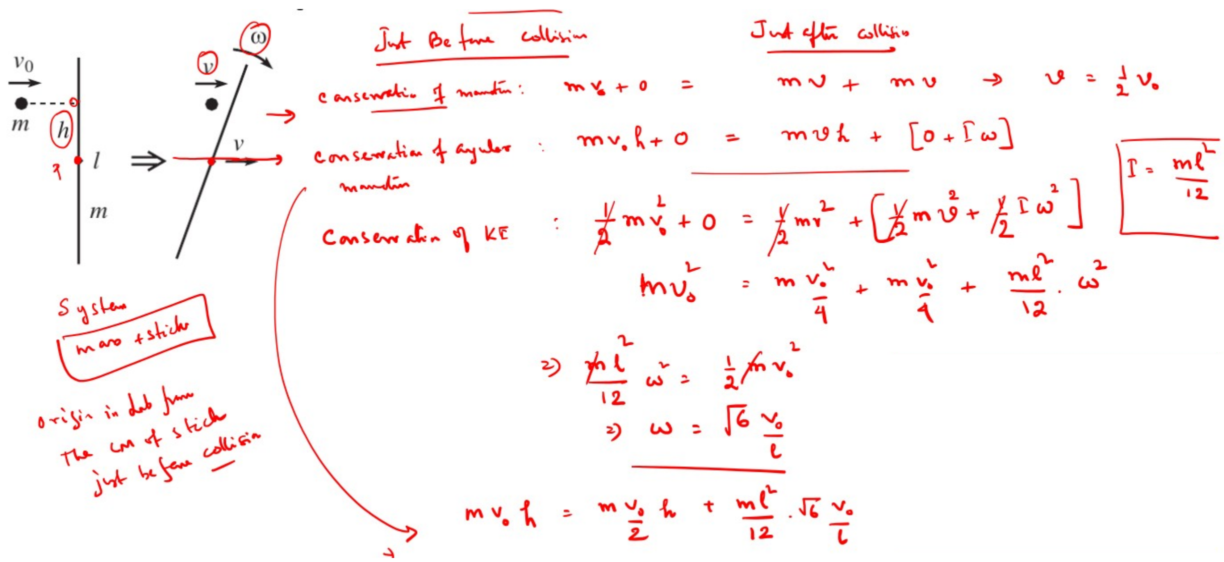
So by symmetry the center of mass is the middle point of the stick. So let us analyze this problem. So, again in collision, as we did in the collision problems, our system is going to be the mass+stick. And since we involves rotation, we must mention the frame of reference. So in our laboratory frame of reference, let us take this point as the origin.

So origin in the left frame is the CM of the stick just before collision. So, then before collision, that is just before collision, and just after collision, now we are going to look at the motion of the system. Now in this case, it is clear that there is no external force acting during the duration of the collision. So we are going to ignore the effect of gravity in this problem. Just for the duration of the collision, if the collision time is very small, we can ignore the effect of gravity.

This is our assumption. So, then the conservation of momentum, and in any case, if we apply the conservation of momentum in the horizontal direction, there is no force in the horizontal direction. Also there is no torque, because if we take this as our origin, at the moment of hitting, the line of action of the gravity will pass through the stick, through the origin. So there will be no torque. So the momentum will be conserved, the angular momentum will be conserved, and since this is an elastic collision, the kinetic energy of the system will be conserved. So these are the three laws that we are going to use to analyze the problem.

So, note that there are three unknowns in this problem. h, where the mass is going to hitt, the velocity v after the collision, and the rotational angular speed ω after the collision of the stick. So just before the collision, so if I apply conservation of momentum, we get the momentum of the system is mv_0+0, because the stick is at rest, and the mass is moving with the velocity v_0. After the collision, the both are starts to move. Now the stick in the x-direction, in this direction, so the mv is the momentum of the mass, and the stick, so we have to now look for the conservation of momentum, look at the center of mass. The center of mass of the stick is moving with a velocity v, and if I apply, then they are equal.

So this gives us the velocity, which is (1/2)v_0. Now what about the energy? Now let us apply the conservation of angular momentum. So what we get is, with respect to this origin, the point mass, even though it is moving in a straight line, translation still have a angular momentum, because the definition R × P. So the angular momentum is mv_{0h}, the stick was at rest, its angular momentum is 0. After the collision, the angular momentum of the stick, of the point mass is, let us say, mvh. So what is the angular momentum of the stick? So we are going to use that this is the angular momentum of the center of mass+the angular momentum about the center of mass.



Now the center of mass is moving in, along this line, and this line is passing through the origin. So the angular momentum of the center of mass about this point is 0. Now, the angular momentum about center of mass is Iω, where this ω is clearly in the along axis, which is axis of rotation is passing to the center of mass, middle point of the stick, and perpendicular to the screen. So, we know from our, this example that this I =$mL^2/12$. Now the last is the conservation of kinetic energy, because it is a drastic collision.

So before the collision, just before the kinetic energy of the mass is (1/2)mv^2, and the stick was at rest. And after the collision, the kinetic energy of the mass is (1/2)mv^2, and the kinetic energy of the stick is again the two parts, as we follow our rule, that as if the whole mass of the stick is concentrated at the center of mass, which is moving to the right with some velocity of the center of mass, so this is (1/2)mv^2+ the kinetic energy about the center of mass, which is (1/2)$I\omega^2$. Now, we solve this and get h.

So, now we have come to the end of this course, so to summarize today's lecture, what we did is that we considered some example situations which involves translation and rotation together, and we also took an example of a collision problem, so we generalized the collision problem to include conservation of angular momentum. So I hope that you have enjoyed this course and learn something about nature and also as this course helped you to increase your level of understanding about the mechanics problem, and most of all I hope that this course gives you some strategy to think about mechanics problem, so that you have now confidence to solve mechanics problem on your own. Thank You. Thank you very much.