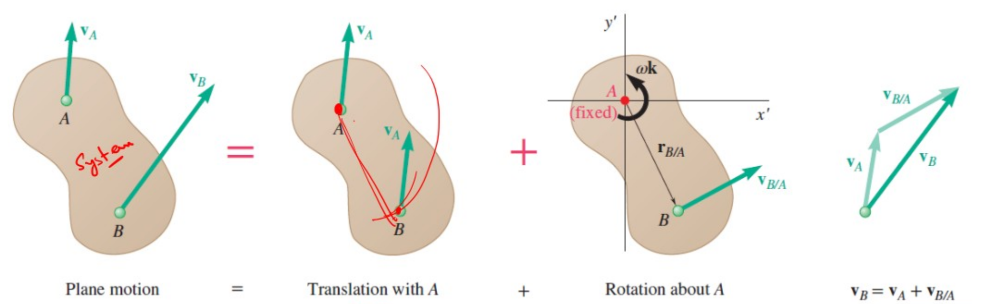
Newtonian Mechanics With Examples

Shiladitya Sengupta
Department of Physics
IIT Roorkee
Week -08
Lecture-43

In the last couple of lectures, we have reviewed the basic concepts that is required to analyze the problems with rotational motion of extended objects or rigid bodies. So, now it is time to consider some practical situations. So today we are going to talk about situations which involves both translation and rotational motion. So these things, situations are usually kind of confusing sometimes, the motion can become really complicated. So, we shall confine our self to not the most general kind of motion, but a special case in which the, what is called the plane motion of rigid body. What does it mean? It means we will consider this special case where the direction of angular velocity is constant or fixed.

That is the angular velocity can change in magnitude, but its direction will not change. So we are going to assume this is the scenario. This basically means that our, so in the rotational part of the motion, the axis of rotation is not changing itself with time. So now in order to, our goal is to develop a systematic strategy to analyze combination of translation and rotation.



So we are going to talk about some computing rules for different quantities one by one. So our basic strategy is the following. So suppose this is an object which is undergoing a combination of plane motion, that is a combination of translation and rotation. So what we are going to do is that we are going to divide the motion into two parts. So, let us say this is our system, so this is our system, and we are going to divide the motion into two parts.

So let us say that our first goal is to answer this question, how to describe the velocity of the such object. So the first part, so we are going to put a pivot point inside the object. So let us say this point is A, so this is our pivot point on the object. The natural choice of this point as we shall see is the center of mass of the object, but we will also take example that we do not have to take center of mass, it can be any point in the object. Then it is always possible to describe let us say any other point B, the motion of B with respect to our lap frame or a frame, inertial frame with fixed origin in which this, so this is the frame in which we are observing this motion of this object, and the object is doing some combination of translation and rotation.

So it is always possible to break the motion of any other point B as a translation of the point A plus rotation about the point A. So a translation of the point A and a rotation of the point B about A, so this is going to be our strategy. Then the velocity of this point B will be the velocity of the point A due to this translation plus the relative velocity which is denoted by this B/A notation, so relative velocity of B with respect to A. So, we have two pieces, we have divided the velocity of the point B which is this velocity B with respect to the lap frame in which the motion is complicated. Then we are divided into two parts, two simple parts, one is that the reference point A, translation of the reference point A which is supposed to be simple to understand, and then the rotation.

Computing rules: velocity

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Velocity of point B In the body w.r.t. lab frame

Translation of reference point A attached to the body

Rotation of point B about ref. point A

So, if you put A as a center of a circle and then take the distance A to B as a radius, and then this is doing a rotation, a fixed axis rotation about the point A. The B is to moving in a circle with a about A with a, so this is the second part, so this is also something simple. So motion in a straight line or translation of a point and rotation about the point. Now, for the case in which we are considering that the axis of rotation is fixed, in that case the translation of the rotation part, the velocity of the point B, the rotation of B about A is given by

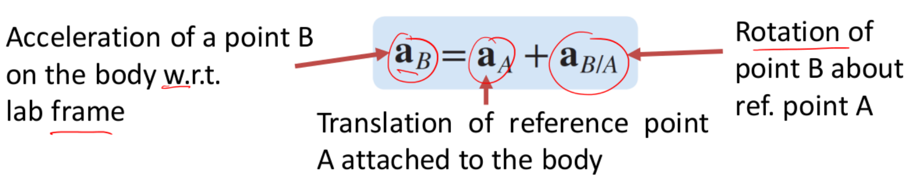
$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} \qquad v_{B/A} = r\omega$$

So angular velocity of the object which is same, so the angular velocity of a rigid body in plane motion is independent of reference point.

If the body is rigid and if it is rotating, then all the point must have the same angular velocity, otherwise the body cannot be rigid. So, this part is same and this is independent of the reference point, and this part denotes the location of the point B with respect to A. So if we take the magnitude and this is a×product that the magnitude of, so note the notation, so the bold space means a vector and without bold means scalar, so this is the magnitude and this is the full vector. The crucial point is that this reference point A can be arbitrary, so this we shall see, so this is our choice and so we shall use, we try to choose a point which makes the problem really simple. So this is our strategy and we can do the same thing with the acceleration.

So, again we are considering for simplicity a situation in which the axis of rotation is fixed. So, this is a plane motion, so the direction of angular velocity is not changing. In that case again, so this body, the object can move with an acceleration, so it does not need to move only with a constant velocity, it can have acceleration. For example, if we talk about a sphere rolling down a plane, it is definitely accelerating. So in that case again, we choose a reference point A which can be arbitrary and then our strategy will be that this motion is a translation of the point A and the rotation about A.

So this is the rule for computing the acceleration of the point B. Again, so a_B is the desired acceleration that you want to compute with respect to the lab frame, the acceleration due to translation of the reference point A plus the acceleration of the point B due to rotation of the point B about reference point A.



So note that the, even though the direction of the magnitude, the direction of the angular velocity is not changing, but this magnitude can change, hence there could be acceleration of the point B with respect to A. So, I just mentioned that the expression for this relative acceleration of the point B with respect to A, it has two pieces. The one piece which is directed along, so this is, so kind of you consider a circular motion of, so this is point A, this is point B, this is the location of the point B with respect to A.

Now if this moves in a circle, as you know there is direction is continuously changing, so it must feel a acceleration towards the center of the circle, that is towards A, which is this term, that is the centripetal acceleration. But if the magnitude of the particle is also changing, then it can also have in general an acceleration in the tangential direction as well, and this is coming from this term. So here this α is the angular acceleration. So the α is equal to the rate of change of ω, that is this is dω/dt. And these are the different pieces, so these are the two pieces in general.

$$\vec{\alpha} = \dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt}$$

So this is easy to prove in spherical, using polar coordinates, in 2D polar coordinates, but here I just state them without proof.

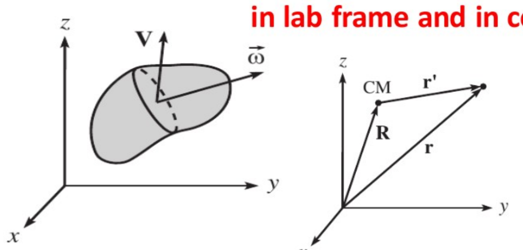
$$\mathbf{a}_B = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$$

$$(\mathbf{a}_{B/A})_t = \alpha\mathbf{k} \times \mathbf{r}_{B/A} \qquad (a_{B/A})_t = r\alpha$$

$$(\mathbf{a}_{B/A})_n = -\omega^2\mathbf{r}_{B/A} \qquad (a_{B/A})_n = r\omega^2$$

Next, angular momentum. Now this is again a strategy, the same strategy we are going to use. So see this object is moving, and its angular momentum will have again two piece, we want to compute the angular momentum in the laboratory frame, the lab frame. Now in this case for a very general setting it is true, not only for the case where the angular velocity is, direction of angular velocity is fixed, but also in a case where the direction of angular velocity can also change, but in most generally there is a theorem which is called the Chasle's theorem, which says that the angular momentum relative to the origin in our lab frame, in which, which is a inertial frame in which Newton's second law of motion is valid, Newton's first law of motion is valid, and in which is fixed origin frame. So we want to compute the angular momentum of the object in that frame, so that has again two pieces.

in lab frame and in center of mass (CM) frame



So look at the center of mass, so now we are specializing to center of mass frame, because we shall see in many different cases, out of all the possible reference points, center of mass is often the simplest choice of the reference point. So, treat the body like a point mass located at the center of mass, and then that may have some motion, may be translational motion that can generate some angular momentum about our origin of the lab frame, plus the angular momentum of the body relative to the center of mass.

L **in the lab frame about the origin is**

$$\mathbf{L} = M(\mathbf{R} \times \mathbf{V}) + \mathbf{L}_{\mathrm{CM}}$$

$$= \mathbf{R} \times \mathbf{P} + \left(I_z^{\mathrm{CM}} \omega'\right) \hat{\mathbf{z}}.$$

Again the same strategy, we are dividing this complicated motion into two simple pieces. The first thing is that, think of a point, the holes or extended object is like a point mass concentrated at the center of mass, and this is moving translation of that point, plus the rotation about the center of mass which generates some angular momentum relative to the center of mass. So in equation, so the angular momentum in the lab frame will have these two pieces, so the first piece, so let us say this is our origin in the lab frame, and some point inside the body which is in the lab frame, this motion of this point is complicated, it has some translation and some rotation.

So, what we are going to do is that, this is the center of mass of the object, and the location of the center of mass is R in our lab frame, and location of the point is smaller, in the lab frame. Then the motion of this point, the angular momentum of this point, is the angular momentum of this point about the center of mass. So if we put center of mass as our pivot point, calculate the angular momentum which is L_{CM}. If the center of mass itself has some velocity, then it will generate some angular momentum given by this expression M(R×V) or R×P, where P is the center of mass momentum. So, just to denote the symbols, so M represents the total mass of the object, R is the center of mass position, note that M is a without bold. So this is a scalar, not bold, R and V are vector, so they are marked denoted in bold face.

And the center of mass, if ω' is the angular velocity about center of mass, then we can, we learn that the angular momentum is the I×ω, where I is the moment of inertia. If it is a fixed axis rotation, simple rotation about a circle, about the center of mass, then our I×ω is simply, has only, so the ω and L will be in the same direction in this particular case, and in that case, we can simply calculate the moment of inertia component along the appropriate component times the angular velocity.

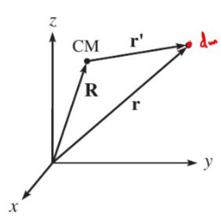
After center of mass, so far we have not talked about energy, but if there is a rotational motion, it also generates some velocity, so it also generates some sort of kinetic energy. So, we should, in order to analyze the problems, we need some expression to calculate the kinetic energy due to rotation. Again we use the same strategy, so our theorem says the kinetic energy of the object will be such that it will have two pieces, as if the whole object has a point mass located at the center of mass location + the kinetic energy due to rotation, pure rotation about the center of mass, pure rotation about the center of mass.

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\boldsymbol{\omega}' \cdot \mathbf{L}_{\mathrm{CM}}$$

So, this is the two pieces, the first piece denote the movement, motion of the center of mass, a point mass with mass M with velocity V located at the center of mass. So let us see how the second part comes in. So here, so we start, so in this case I am using T to denote kinetic energy, so note that kinetic energy is a scalar, the T is without boldface, but velocity is a vector, so it is with boldface. So now we know that suppose this point has some velocity v, which is a combination of this total velocities in the lab frame, so it is a combination of all the complicated motion resulted in a velocity v of this point, let us say this is a small mass element dm. Then its kinetic energy is always

in lab frame and in center of mass (CM) frame

$$\begin{aligned} T &= \int \frac{1}{2}v^2\,dm \\ &= \int \frac{1}{2}|\mathbf{V} + \mathbf{v}'|^2\,dm \\ &= \frac{1}{2}\int V^2\,dm + \frac{1}{2}\int v'^2\,dm \\ &= \frac{1}{2}MV^2 + \frac{1}{2}\int r'^2\omega'^2 dm \\ &= \frac{1}{2}MV^2 + \frac{1}{2}I_z^{\mathrm{CM}}\omega'^2. \end{aligned}$$



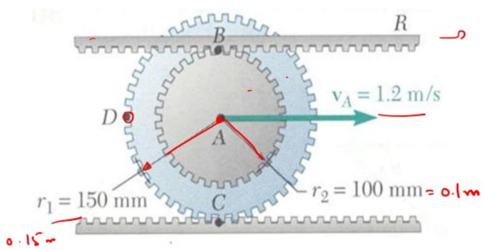
$$\vec{r} = \vec{R} + \vec{r}'$$
$$\frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt}$$
$$\vec{v} = \vec{V} + \vec{v}'$$

kinetic energy *T* in the lab frame is

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\boldsymbol{\omega}' \cdot \mathbf{L}_{\mathrm{CM}}$$

Now this v has two pieces, so the first piece, so the this is the V, the first piece come. So, r from this picture is R+r'. So if I take all the derivative in the lab frame will be, so this is our v, and this is V which is the velocity at the center of mass, and this is v' which is the velocity of the point with respect to the center of mass. So, then our v is $|V+v'|^2$, and then if you expand this square, do the algebra, you end up with this expression that this kinetic energy of the object in the lab frame has two piece, first piece is due to the, as if the whole mass is concentrated, whole object mass is concentrated at the center of mass, and this moving with the velocity of the center of mass. And the second piece is turns out to be of the form the moment of inertia times ω'^2, so this is the expression for kinetic energy due to rotation, so this piece is the kinetic due to rotation about center of mass, and you can also using L = I ω, you can also write this as in this particular form ω L, so dot product between the vectors ω' which is the angular velocity in the center of mass frame dot product and the angular momentum in the center of mass frame.

So, the other notations are explained already. So, let us now apply these rules to systematically to solve problems, because our goal is always to sort of develop a method to solve problems. So, let us look at this problem. So, our first problem is the following.



The double gear rolls on the stationary lower rack: the velocity of its center A is 1.2 m/s. Determine

a) the angular velocity of the gear, and
b) the velocities of the upper rack R and point D of the gear.

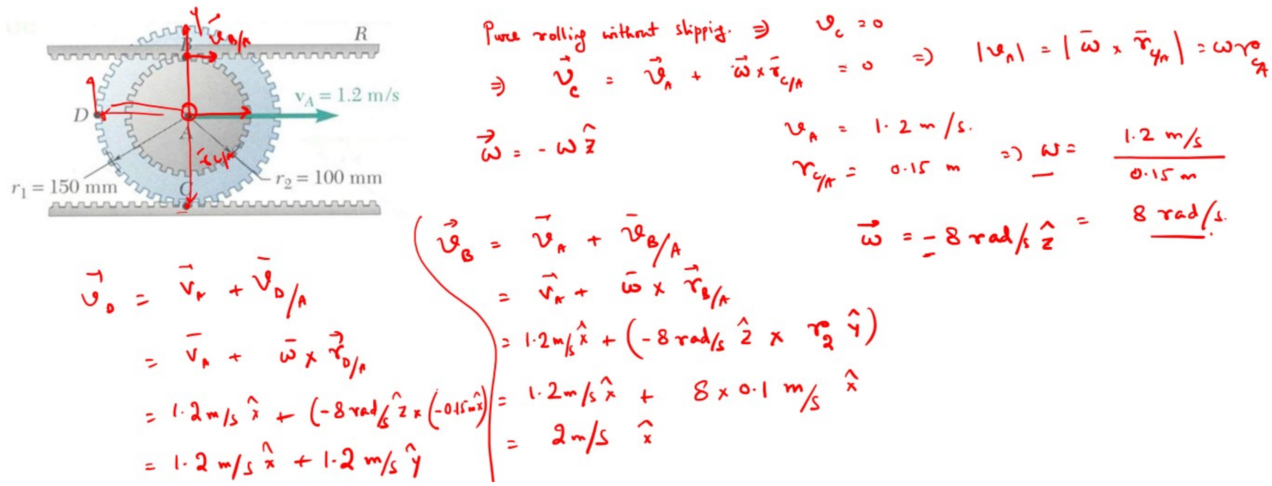
Ans: (a) $\boldsymbol{\omega} = \omega \mathbf{k} = -8$ rad / s (b) $\mathbf{v}_R = 2$ m/s $\mathbf{i}$, $\mathbf{v}_D = 1.2$ m/s $\mathbf{i} + 1.2$ m/s $\mathbf{j}$

Image ref: *Vector Mechanics for Engineers,* by F. Beer, E. Johnston, D. Mazurek, P. Cornwell, B Self, McGraw Hill.

So we have a rolling gear, so this is a picture of a gear, so it is a double gear, so 1 and 2, so grey and blue, two gears, and they are moving in such a way that the, so this is a rack on the lower side which is at rest. Now this gear mechanism works in such a way that the center of the gear A is moving to the right with some velocity v_A. So, now the size that is the radius of the low smaller gear is given which is r_2 and the bigger gear is r_1.

So, how to solve this problem? Again our strategy is the following, so we simplify this picture like a two circle, two concentric circle, then this is our system, so this is our system, and now in rotation the after specifying the system we will also need to specify the reference frame, what is the lab frame. So, let us say a lab frame is a fixed frame in which this whole thing is moving, so we are going to describe the complicated motion of this gear as the translation. So, we will take this center point A as our reference point, so this reference point is moving with some velocity v_A, so this is the translation piece, and every point on this gear is rotating about A, so that is going to be our rotation piece. So, if we combine these two piece we get the full motion of any point on the gear with respect to the lab frame. So, this is our so called rolling motion of a wheel, so this is our strategy.

Now let us calculate the numbers, so the first thing is that if the, the roll of the wheel gear, so first thing to note that we are going to assume this condition that this is a pure rolling without slipping, that means that the point of contact, the instantaneous point of contact of the gear wheel with this lower rack must be 0, because it is given that this lower rack is not moving, so that means the point of contact of C must be 0. Then we can see that this basically give a constraint that the v_A, so v_C has two pieces, one is $v_A + v_{C/A}$ which is $\omega \times r_{C/A}$ and that is 0, that means the magnitude of v_A must be equal to magnitude of $\omega \times r_{C/A}$. It is the location of C with respect to A, so this is simply ω. So, the ω is clear from this picture if we apply right hand thumb rule that let us say this is our x-axis, so let us say this is our x-axis, this is our y-axis, then the ω must be in the some z-direction,



So, now we want to find out the velocity of the upper rack R and the point D of the gear. So, the upper rack R has the same velocity as the point of contact between the this low, smaller gear and the upper rack. So, v_B the is going to be $v_A + v_{B/A}$.

So, you can see that the net velocity in the left frame of point D will be this direction it will have two components, this component horizontal component is due to the translational motion of point A, and this component is due to the rotational motion of about the point A. So, to summarize today we did sort of looked at a situation where there is a combination of translation and rotation, and we learnt how to systematically approach those kind of problems. We shall going to take more example in the next lecture. Thank you.