Newtonian Mechanics With Examples

Shiladitya Sengupta
Department of Physics
IIT Roorkee
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Lecture-42

Let us continue our discussion of rotational motion. In the last lecture, we saw the relation between the angular velocity and angular momentum. And we sort of learnt that there is a strange fact that angular momentum and angular velocity need not be in the same direction. So, this is very different from the situation in the case of translational motion where the linear momentum P is equal to mv is always in the same direction as the linear velocity v. Today, our goal is to examine the cause of rotation. So, when we discussed Newton's law of motion, so we said that the cause of motion is the, is the force.

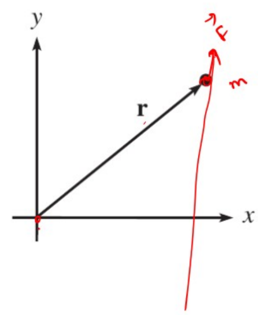
So, that was a confusion for long time among people that the, what is the origin of motion. Sometimes people thought that the origin of motion is the change in position. So, the velocity is, if you have a velocity that means the, that is the cause of, so the force is some external agent that causes the, some object to move and for long time people had this misconception that force causes change in the position, that is velocity. But later when the Newton's, after Newton's law of motion by the research of Galileo, Newton, etc., it is now clear that force does not, that is the interaction between our system with its surrounding causes acceleration, that is change in velocity, not change in position.

Now, today our goal is to look at what is the analogous situation for the rotation, what causes rotation. Now this we have implicitly, we know that what causes rotation is the torque. So, that is another consequence of force. So force on a, so if we take a system and the system, so it is interacting with its surrounding, so there are some, these interactions of the forces and this can cause our system to translate, but another consequence of the force is the rotation.

It can also cause our system to rotate. This is why we not only need to mention the magnitude and direction of the forces acting, but also the line of action of the force. So this is what we learnt when we learnt how to draw the free body diagram correctly, that you need to specify the line of action also, because the force can also cause rotation. So today is our goal is to examine the relation between force and its rotating effect, which is expressed by torque. So, what is the relation between angular momentum and torque? So you know from your high school physics that the two things, that this torque is expressed as the definition of torque is that this is the moment of force and this is expressed as a cross product.

So let us say that we consider a single point mass and with respect to some particular origin and this origin is fixed in space and this is our coordinate system. So, this vector R is the position vector of this point mass and let us say some force is acting, let us say in this direction. Then so this is the line of action of the force. Then the torque due to this force about this point pivot point that is the origin is given by this definition r×A. And you have also seen that this causes the effect of this force is to change the angular momentum, which is analogous to the Newton's law that the cause of force is to change is equal to the rate of change of linear momentum.

Point mass, fixed origin



So, our goal is today to critically examine this relation. So, remember that this relation we also used implicitly when we talked about the statics problems when the objects are motionless and we use this torque balance condition as one of the essential criteria for statics or mechanical equilibrium. In that case we used this definition always that τ=r×F. Today we are going to see the relation between the angular momentum and the torque. And specifically we are we want to discuss this question that under what condition τ=r×F .

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{L}}{dt}$$

$$\bar{L} = \bar{r} \times \bar{p}$$

$$\frac{d\bar{L}}{dt} = \frac{d}{dt}(\bar{r} \times \bar{p})$$

$$= \underbrace{\frac{d\bar{r}}{dt} \times \bar{p}}_{\substack{\bar{v} \times m\bar{v} \\ =0}} + \bar{r} \times \frac{d\bar{p}}{dt}$$

$$\boxed{\frac{d\bar{L}}{dt} = \vec{r} \times \bar{F}}$$

So, when I say torque I mean that some external agent that causing a change in the angular momentum. So, let us work it out in some detail. So, let us say that we start with our definition that for a point mass the angular momentum of this particle of mass m is given by L=r×p. So then if we look at the rate of change of angular momentum that will be the rate of change of this quantity. Then it has two terms, so I can apply the chain rule of taking derivatives.

Now this is v and this is mv. So, the first term is a is basically a cross product of velocity with respect to itself apart from the mass. That means this is 0 by definition of the cross product. Hence we have the dL/dt which is equal to r×(dp/dt) which is the force as listed here. So this is how we get this relation for a single point mass.

Now before proceeding further let me discuss a very interesting and very important consequence of this particular case of a point mass. So, we are talking about the central force motion. That is a particle is subjected to a central force. What is a central force? It is a force such that if this is the particle, so let us say some particle is moving along the origin and there is a force acting on this particle which is along the its position vector always. So, this is the, so the direction of the central force is along the position vector.

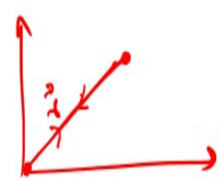
So this is the definition of a central force, that is why it is called central. Now it can be towards the origin or it can be away from the origin. So the F(r) can be positive or negative, the magnitude of the force, but the direction is always along the position vector. So, if this is the case then we can immediately see that a particle of moving in such condition will have a constant angular momentum.

If a particle is subject to a central force only, then

Theorem 1: its angular momentum is *conserved*.

$$\text{If } V(\mathbf{r}) = V(r), \quad \text{then } \underline{\frac{d\mathbf{L}}{dt} = 0}.$$

i.e. if $\mathbf{F} \propto \mathbf{r}$.



$$\frac{d\bar{L}}{dt} = \bar{r} \times \bar{F}$$
$$= \bar{r} \times F(r)\,\hat{r}$$
$$= 0$$

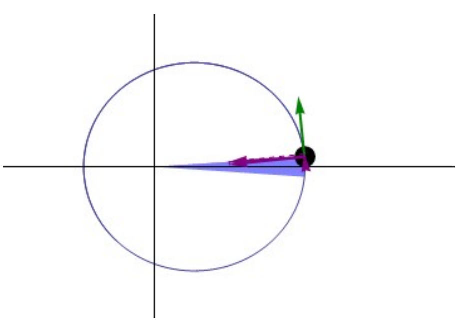
$\bar{L}$ = <u>conserved</u>

Now if F is along the direction of position vector then this is 0. So, then if $F \propto r$, then dL/dt=0, that means that L is conserved, that is the particle's angular momentum is a constant. And geometrically if the angular momentum is constant that the only way it is possible if the particle is moving in a plane which is perpendicular to the direction of the angular momentum. That is the only way the angular momentum can remain constant. You can check that.

Theorem 2: its motion takes place in a *plane*.

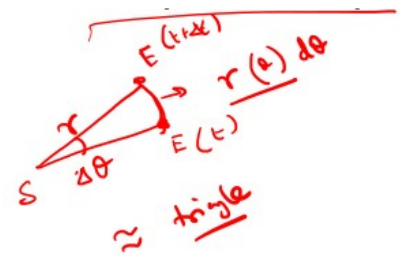
So here is a very, very important application of this simple concepts. So we take this example of Kepler's laws of planetary motion. Using just this simple definition we can understand Kepler's laws of planetary motion. So let us start with Kepler's second law. So, what does it say? So, basically in this case the origin is for could be a massive particle such as sun and this particle that is moving around the origin is let us say our earth or any planet.

So, the Kepler's second law says that a line segment, so here there is a line segment. So, the violet line joining the particle and the center that is the sun, sweeps out equal areas during equal intervals of time. So, this is the blue shaded region. So, notice that the particle is not moving in a constant speed. Sometimes it is moving fast, sometimes it is moving slow and you can see that if you follow the blue shaded triangle sometimes it becomes wider, sometimes it is narrower.



Why because this is showing that for equal amount of time if the speed is higher then it will cover more angular earth. So, the blue shaded region will appear wider. Whereas if the speed is small then the arc length will be narrow and the triangle will appear narrower. But Kepler's law says that the area of this triangle of the blue shaded region for equal amount of time Δt is always the same.

So let us derive this. It is very easy to derive this. So, the blue shaded region, blue shaded triangle, so let us draw some particular instant. Let us say it makes an angle, so this is our sun and this is at the earth's position at t and this is the earth's position at t + Δt. And this angle is Δθ . Then the arc length, this arc length is r times, so r which is, so in general the distance between sun and earth are different.



So the r which is the distance between sun and earth is a function of θ . And now you can, if Δθ is very small then you can approximate this piece of the, this arc as a straight line so that this is approximately a triangle. So you can use the formula for the area of a triangle half into base into height. So, this is base and this is height and so this is r and you get the half times r and dθ , if the speed, angular speed is ω then the angle covered Δθ is, so this is your Δθ will be ωdt. So then if you take the rate of change then dA/dt will be (1/2)r rω and rω is the tangential speed of the earth along the trajectory.

$$
\begin{aligned}
dA &= \frac{1}{2}\underbrace{r(\theta)}_{\text{base}}\,\underbrace{r(\theta)d\theta}_{\text{height}} \\
&= \frac{1}{2}r(\theta)\,r(\theta)\underbrace{\omega\,dt}_{\to\,\Delta\theta} \\
\frac{dA}{dt} &= \frac{1}{2}r(\theta)\;[\underbrace{r\,\omega}_{v}] \\
&= \frac{1}{2}r\,v \qquad \Rightarrow \frac{dA}{dt} = \text{const} \\
&= \frac{L \;\to\; \text{constant}}{2m \;\to\; \text{constant}}
\end{aligned}
$$

$$\bar{L} = \bar{r}\times\bar{p} = \bar{r}\times m\bar{v} \Rightarrow |\bar{L}| = mvr \Rightarrow rv = \frac{L}{m}$$

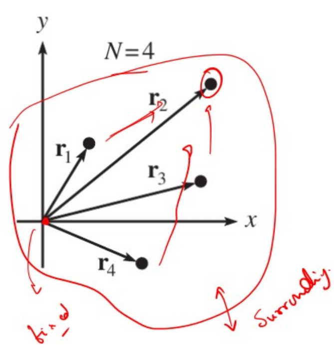
So, this is basically the content of Kepler's second law. I also mentioned the nature of the trajectory.

So, Kepler's first law. So, the first law says that the orbit of a planet is an ellipse with the sun at one of the two foci. Now what from this simple analysis of the conservation of momentum we can only say that the sun must move in the same plane.

So, in general it could have some complicated trajectory but no it always moves in the same plane. So, this is consistent with Kepler's first law because ellipse is an orbit in a plane. So, it basically shows that the center of the sun is always in a same plane. Now we cannot from this analysis we can only say that the orbit is planar. So, Kepler's first law contains more information, it in fact says that the orbit is ellipse.

So this is true only for a special case where the force law is $\propto 1/r^2$. So note that here we just mentioned that this is a some function of r. So when I say some function of r and it is directed, we just assume that this force between sun and earth is directed along the line joining sun and earth. But we do not know what is the magnitude. So, once we know the magnitude, then like the using these Newton's laws of universal gravitation, then Newton was able to prove that the law, this orbit, the planar motion is in fact an ellipse.

So, let us come back to our discussion of under what condition, the torque that is dL/dt is equal to, so this I mean the change in, rate of change in angular momentum is equal to r×F. So now we have, so we consider a single point mass. Now we generalize it to an extended object. So, let us for simplicity we describe this extended object as a collection of point mass. So, the first case that we consider will be where our origin is again fixed.



So, like before this is our origin and this origin is fixed and there are sort of point mass and for concreteness we have four point mass. Now in this case we have to calculate the dL/dt. So, let us work it out. So, in this case L, the total angular momentum, so we can compute it for each of this particle. So, for particle I, position is r_i and its momentum is p_i and then we have to take a sum over all particles.

Then the rate of change of angular momentum is equal to,

$$\bar{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\frac{d\bar{L}}{dt} = \sum_i \frac{d}{dt}\left(\vec{r}_i \times \vec{p}_i\right)$$

$$= \sum_i \left[\underline{\frac{d\vec{r}_i}{dt}} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt}\right]$$

Here is the difference. So, for a, when we have a extended object, then suppose when we write down this, the force on particle i, so we make a distinction between two kinds of force. So, let us say this is our system, so this is our system and everything else is surrounding. Then there will be interaction between the surrounding and the particles inside the system, from surrounding interactions, surrounding from and system, but each particle of the system can also have force from other particles, particles or parts of the system. So we call this as external force or external interaction and this one as internal force or internal interaction. So, then our Newton's law will be the total force, so this is total force on particle i, so this will be as an external component and an internal component.

The origins are different, the external component is coming from outside the surrounding, from the surrounding and internal component comes from the other particles in the system. So, let us say if I take this particle, then the force due to r_1, r_3 and r_4 constitutes the internal force on particle 2. So, we write it as,

$\bar{p}_i = m_i \bar{v}_i$

$$= \sum_i \bar{r}_i \times \frac{d\bar{p}_i}{dt} = \sum_i \bar{r}_i \times \bar{F}_i^{ext} + \overbrace{\sum_i \bar{r}_i \times \cancel{\bar{F}_i^{int}}}^{0}$$

$$\frac{d\bar{L}}{dt} = \sum_i \bar{r}_i \times \bar{F}_i^{ext}$$

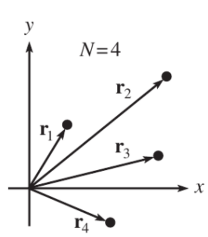
$$\boxed{\frac{d\bar{L}}{dt} = \sum_i \bar{\tau}_i^{ext}}$$

$\frac{d\bar{p}_i}{dt}$ → Surrounding and system, external

→ interaction with other particles/part of the system, internal

$$\frac{d\bar{p}_i}{dt} = \bar{F}_i^{tot} = \bar{F}_i^{ext} + \bar{F}_i^{int}$$

Extended mass, fixed origin



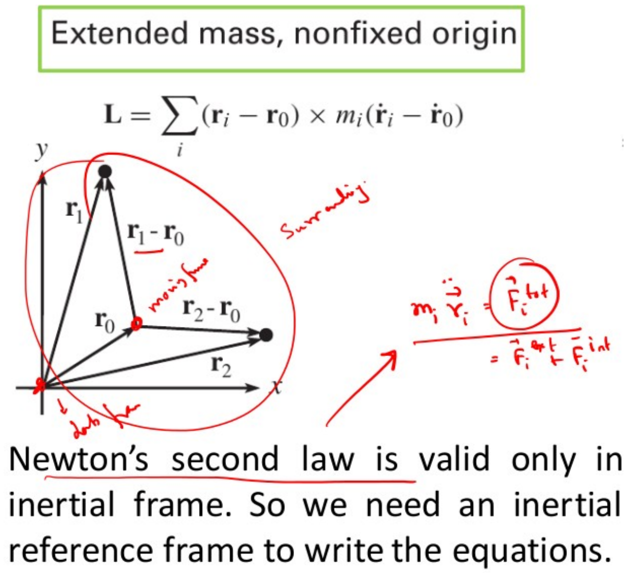
Take home exercise

Zero torque from internal forces: Given a collection of particles with positions $\boldsymbol{r}_i$, let the force on the ith particle due to all the others be $\boldsymbol{F}_i^{int}$. Assume that the force between any two particles is directed along the line between them (central force). Use Newton's third law to show that

$$\underbrace{\sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{int}}_{\tau_{int}} = \boldsymbol{0}$$

$\vec{F}_{12} = -\vec{F}_{21}$

I showed is that all the internal force, the torque if you sum over all the forces, the torque due to all the internal forces will be 0, so the only force that matters are the forces that are coming from the surrounding. And if you recall the problems that we did, the examples that we did, so when we talked about the torque on an object, it was always the torque coming from the surrounding. For example, if I look at the torque due to earth's interaction, so that is a force, if this is our system, then the, what causes the rotation is an external force like the gravitational attraction due to earth, which is coming from the surrounding. So, this is true even for an external force, but here comes the third case and the most interesting case.



So, in this case, we now consider our frame which is moving. So far in the two situations that we considered, the frames where that is our origin was fixed in space. So, these are what is called inertial frames, that is we assume that in frames, these frames, the Newton's second law is valid, which means that I mean, in this is a frame in which Newton's first law holds, that is if there is no other external interaction, then the particles will move with a constant velocity. But now, let us assume that our origin itself is moving with respect to some inertial frame, so and when I say moving, it can have, move with constant velocity or it can have some acceleration, it can be an accelerating frame. So, this is a very interesting situation, which we did not discuss more like earlier in our course, and this is the only situation in which we will briefly mention this, but so this is, this non-inertial or accelerating frame of reference. And suppose, if the question is, if we take such kind of reference where the origin itself is moving, then can we write down the torque is equal to r×F, this is an interesting case.

Now, because we cannot write, this is not guaranteed that in that frame, the Newton's law will, Newton's laws of motion will hold. So, we need a second frame which is fixed, an inertial frame. So, this is the frame which which we will call the lap frame. So, consider that this is the origin of the lap frame, and this is the r_0 is the origin that is moving. The position of the origin in the lap frame is r_0, and in the lap frame, the position of all the particles, so we have only two particles in this particular example is r_1 and r_2.

Now, the position vector of the same particles with respect to this moving frame. So, this is the moving frame, is r_1-r_0 and r_2-r_0. So, we want to calculate that what is the rate of change of angular momentum of this system of two particles in the moving frame. So, the first term is r_i-r_0, this is the position with respect to the moving frame, times the momentum with respect to the moving frame of particle i. So, we have to calculate the time derivative, and as before we get two terms, the first term will be the velocity times momentum in the moving frame, so it is 0, and the second term is the interesting term, which is the position of the particle i in the moving frame, times the derivative of the momentum, and the derivative of the momentum now has two terms.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \underbrace{m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)}_{\text{momentum of particle } i \text{ in moving frame}}\right)$$

$$= \sum_i (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) + \sum_i \underbrace{(\mathbf{r}_i - \mathbf{r}_0)}_{\text{position of } i \text{ in moving frame}} \times m_i(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_0)$$

$$= 0 + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} - m_i\ddot{\mathbf{r}}_0)$$

acceleration of i in moving

$\ddot{\vec{r}}_i$ = acceleration of i in lab frame

$$\sum_i (\vec{r}_i - \vec{r}_0) \times (\vec{F}_i^{ext} + \vec{F}_i^{int})$$

$$- \sum_i (\vec{r}_i - \vec{r}_0) \times m_i \ddot{\vec{r}}_0$$

$-\vec{F}_i^{int}$

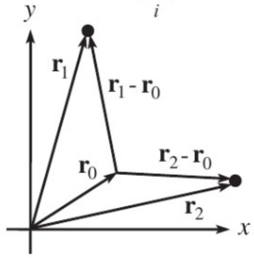
Now, the Newton's law says the m_i acceleration is equal to the total force on particle i, this note that you can write only in the fixed frame, it is guaranteed only in the fixed frame, there is no way you can, it is guaranteed if the frame is moving. If the frame is accelerating, this is definitely not true. So note that this is the real physical interaction which obeys the Newton's third law, so we divide as before into external force and internal force, so in this particular example the force on 1 by 2 is the internal force, and everything else is in the surrounding, so this is our system, everything else is in the surrounding.

So I am writing all the equation with respect to the fixed frame. Now as before you can show that the first term is, so this is the sum over i all the external torque. The second term with the same argument you can show by applying Newton's third law, this is going to be 0.

Now the last term, so the last term is, in this case, if I recall the center of mass definition, so this is nothing but the total mass of the system times the position of the center of mass. So, we see that the dL/dt, this is not only the τ external as before, but has some extra term.

Extended mass, nonfixed origin

$$\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)$$



$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)\right)$$

$$= 0 + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} - m_i\ddot{\mathbf{r}}_0)$$

$$\sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i^{ext} = \sum_i \vec{\tau}_i^{ext}$$

$$\sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i^{int} = 0$$

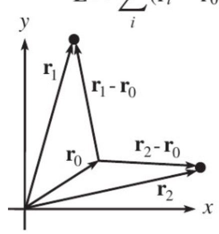
$$-\sum_i m_i (\vec{r}_i - \vec{r}_0) \times \ddot{\vec{r}}_0 = -\left[\left(\sum_i m_i \vec{r}_i\right) \times \ddot{\vec{r}}_0 - \left(\sum_i m_i\right) \vec{r}_0 \times \ddot{\vec{r}}_0\right]$$

$$= -\left[M\vec{R} \times \ddot{\vec{r}}_0 - M\vec{r}_0 \times \ddot{\vec{r}}_0\right]$$

And only if this extra term is 0, then we get this relation that dL/dt is equal to r×F. So, this is the r×F term. So, under what condition the second term is 0. So, you see from looking at the second term, that if this is the center of mass position. So, if the moving point is the center of mass, that is the origin is the center of mass, then this term is 0, that is $r=r_0$.

Extended mass, nonfixed origin

$$\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)$$



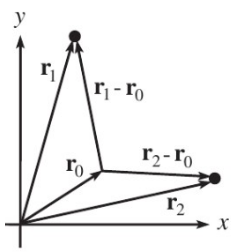
$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)\right)$$

$$= \left(\sum_i \underbrace{(\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}}}_{\vec{\tau}_i^{ext}}\right) - M(\mathbf{R} - \mathbf{r}_0) \times \ddot{\mathbf{r}}_0$$

Or if this acceleration is 0, then also this term is 0, that the origin is moving, is not, can be center of mass, can be different, but it is moving with a constant velocity, not accelerating with respect to the fixed frame. Or the, this is the, the origin is different from center of mass and is also accelerating. But for some reason, this difference is the parallel to the acceleration. So, this is a very special and strange condition. If one of these three conditions satisfies, then we see that this term can be 0, in that case the cross product itself will be 0.

And we end up with the relation that the rate of change of angular momentum is equal to the rate of the total external torque, that is r×F. So, this is something that is usually not highlighted in textbooks, so I just wanted to mention it. So, to summarize, if you take a inertial frame, then, so we looked at the relation between the rate of change of angular momentum and torque, we found that in a inertial frame, the rate of change of angular momentum is equal to r×F. But if we take a moving frame of reference, it is not true in general and we discuss the conditions under which it is true. Thank you.

Extended mass, nonfixed origin



$$\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)$$

CM position

$$\frac{d\mathbf{L}}{dt} = \left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} \right) - M(\mathbf{R} - \mathbf{r}_0) \times \ddot{\mathbf{r}}_0$$

1. $\mathbf{R} = \mathbf{r}_0$, that is, the origin is the CM.
2. $\ddot{\mathbf{r}}_0 = \mathbf{0}$, that is, the origin is not accelerating.
3. $(\mathbf{R} - \mathbf{r}_0)$ is parallel to $\ddot{\mathbf{r}}_0$.

$$\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} \equiv \sum_i \boldsymbol{\tau}_i^{\text{ext}}$$