Course Name: Newtonian Mechanics With Examples
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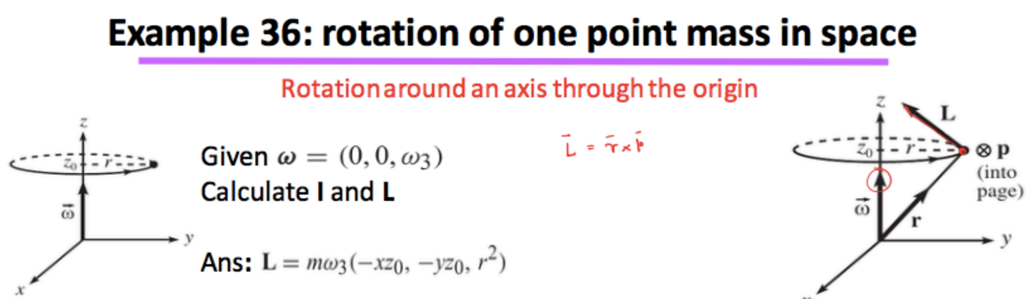
Week 08
Lecture – 41

Hello everyone! Let us continue our discussion of the translation and rotation of rigid bodies. A few lectures ago, I mentioned a strange fact about the rotation of an extended object. So, if we take a translation, let us say a point particle, it goes from somewhere here to here. So you can go. Let us say this is along the x axis. You take two steps, and then let us say you take three steps to reach your destination.

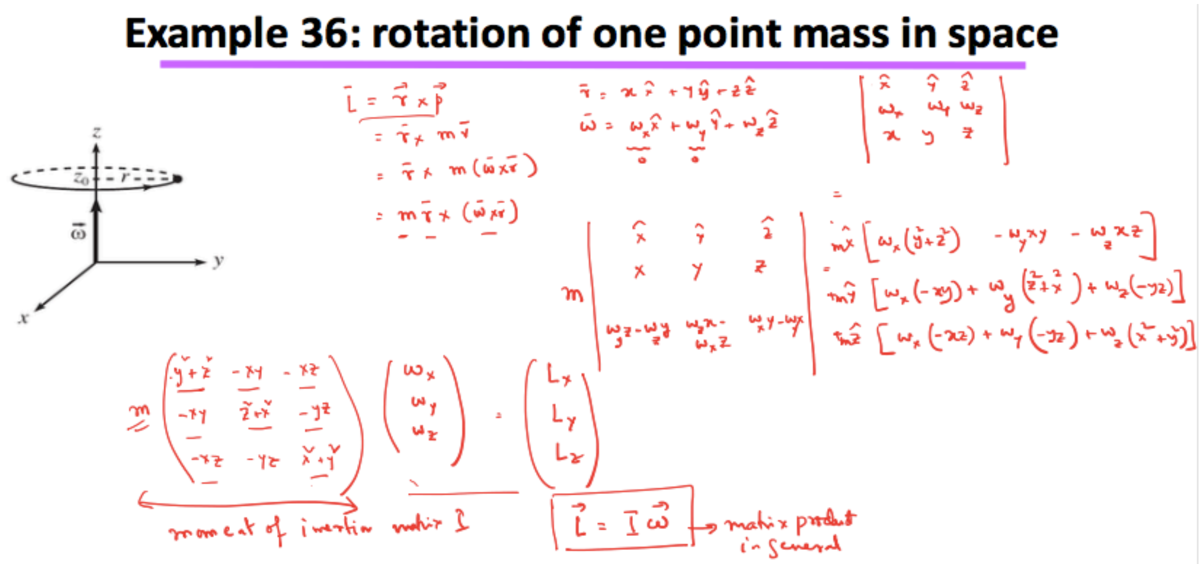


Now you can go first along the x axis, two steps, and then along the y axis, three steps. Or you can take three step along y axis first and then take two step along x axis, you will reach the same final destination. However, this is different with the case of rotation. So, let us say if we take this as our extended object and suppose this is our x axis.

This is our y axis and remember we mentioned the strange fact that if I rotate by some finite angle such as, let us say, 90 degree, let us say, first about the x axis and then about the y axis, the object orientation will become like this. However, if I first rotate by 90 degree along the y axis and then by 90 degree along the x axis,. We get a different orientation. So, the first strange fact about rotation was that the rotation by a finite angle about some axis does not commute. Hence, when we talk about rotation, we always talk about infinitesimal, very small in the limit of the change in the angle tending to 0, infinitesimal rotation.

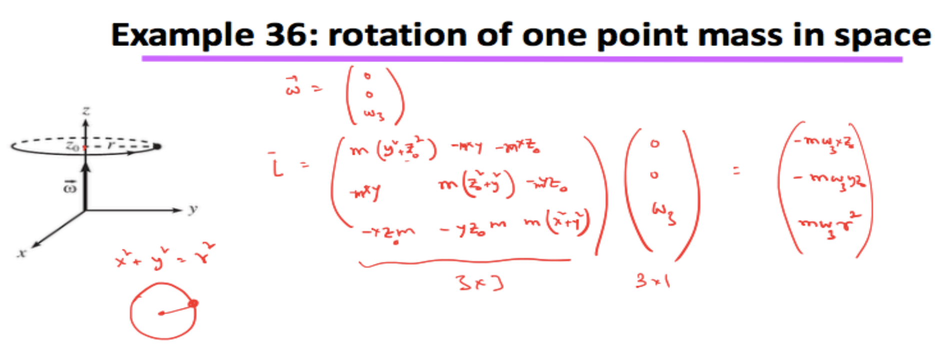
And more precisely we talk about the rate of change of angle or the angular velocity. So, this is what we discussed in the last lecture about angular velocity and we reviewed examined in detail about the vector nature of the angular velocity. In today's lecture, we are going to talk about another very counterintuitive property of rotation. So, which is why the rotation is somewhat, students find somewhat stranger compared to translational motion. So, I am going to talk about the fact, so today our goal is to look at the relation between the angular velocity and angular momentum.

So, in the case of linear momentum and linear velocity, we know that if a particle has a mass m and moving with a velocity v, then its linear momentum is given by this particular formula when the velocity is very small compared to the speed of light. So, the main point for today's purpose is that mass is a scalar. So, the direction of the momentum and the direction of the velocity are always same. Usually the magnitudes are different because of the mass, but the directions are always the same. So this is a translation.



Today we are going to look at the analogous formula for rotational motion. So, what is the relation between the angular velocity omega and angular momentum L? And here is the strange fact that we want to highlight at the beginning is that the angular momentum L and angular velocity omega need not be in the same direction. So, this is the first counterintuitive, very strange fact. So let us understand this by working out an example. So consider rotation around an axis through an origin.

So let us say our origin is here, and there is a point particle of mass m, just a single point particle, a point mass. It is rotating in space along this particular circle. The center of the circle is on the z axis. So, that is how we choose our reference frame, such that the center of the circle lies on the z axis at a height of z0 from the origin. And we are free to choose the x and y axis, and these are chosen accordingly.



Now, given that this angular speed is omega3, it is a speed and the direction of the angular velocity is along the z axis; this is a constant. So, the particle is moving with a constant angular speed omega3 and the direction of the angular velocity is pointed along z axis, which means that the particle is moving in a plane which is perpendicular to the z axis, which means in a plane which is parallel to the xy plane. Now, the problem is to calculate the angular momentum and calculate the moment of inertia I. So, let us first, before going to the calculation, let me show you pictorially that we shall find that the angular momentum and angular whatever the magnitude, but the directions are different from the angular velocity. So, we know that angular momentum is given by R cross P.

So you see that the P, Suppose you take on some instant and the position of the particle is at this point in the circle. So that the P is going into the pitch is tangential direction of the circle. So this is rotating like this origin is here, so we are looking from this side. So, it is going into the pitch. Now, the position vector of this point mass from the origin is R.

So then the R cross P at this instant is pointed in this direction, whereas the angular velocity is pointed along the z axis. So, it is a strange fact that angular velocity and angular momentum need not be in the same direction. So, let us now work out how it is possible. So let us now work out in detail. So, let us start with the detailed calculation.

So, we start with the fact that angular momentum, so this is the definition of angular momentum. Now we know that, so this is momentum: P times v. Now, this v is again, and we know that this is omega cross R. So you can take this mass on the other side. So, you can write M R cross omega cross R.

Let us work out this double-cross product in component wise. So, let us say, what is R? So R is x, y and z. Omega it is given that this is, so the first cross product is going to be, so these are the components of omega, and these are the components of z. So, what we are going to find is for the first case. So let us take the more general case.

So, let us assume that omega has these components and given that this is 0 and this is 0. But I want to deliberately keep it general to sort of get the structure correctly. So, we are going to assume, so this is the most general case. So, this and then we have the second dot cross product with this term. So, let us write it down here.
So, we just need this to evaluate this particular cross product. So let us work it out. So, the x component will be omega y z minus omega z y. y component will be omega z x minus omega x z and z component will be omega x y minus omega y x. So let us write down the first component.

The first component is y times, so we have y square times omega x minus omega y, xy and then we have minus omega z, xz plus omega x z square. So, I wrote. It was organized in this way to sort of show you some deliberate pattern in the terms. So this first term is omega x times something, the second term is omega y times something, and the third term is omega z times something. And let us see if the same pattern we guess should hold for the other components as well.

So let us check that. For y we have omega x, so for y component, we have omega, so this times this minus this times this, so the omega x terms will contain a minus x z. Then we have omega y. So omega y, the first term contains x square, Sorry, so z times this, there is a z square and the other term has this times this; this will contribute 1 x square. And omega z will have the term y z minus y z. And similarly for the z component, we have omega x, which gives me this times this.

It has x z minus x z, omega y times omega y is coming in this term, so this is minus y z. And finally, omega z has two contribution, one coming from x square and the other coming from y square. So, now we can see that we can, we want to make the pattern more clear, so we want to write it. The pattern will be more clear if we write it as a product of two matrices. So let us say, let me correct this mistake, this should be x y.

So, now we want to write it in a product, so we take this vector, we represent this omega the vector as a column vector. And then we express this as a matrix, so a 3 by 3 matrix. Then we see that the first term should be, second term should be… Now, there is a M sitting over here, so the whole thing has to be multiplied by M. So, there will be a M in each of this term, so there will be M, so I am going to include this M. I am writing it here, so this whole thing, the M, will be part of this matrix.

So, if you recall the definition of moment of inertia that we discussed in the previous lectures. You see that this is precisely the matrix; the diagonal elements are precisely the moment of inertia about x y and z axis, and the off-diagonal elements are the products of inertia that we defined earlier, except for a negative sign. So, we can redefine the product of inertia with a negative sign, If we do that, then This matrix includes this mass M is going to be our moment of inertia matrix I, and This product, from this definition,. This product is going to give us the components of angular momentum. So now let us, this is a general definition, this is a general definition, So what we have? A L is a matrix times a vector, and this is why the mathematical reason that If you multiply a vector with a scalar, then the direction of the product and the direction of the phase vectors are same, but if you multiply a vector with a matrix, then the resultant, you also get a vector, but this resultant vector need not be in the same direction as the original vector.

So, this is a matrix equation in general, even for a point mass it can be a matrix equation. So, this is the reason that L and omega are not in the same direction. So, let us now plug in the values and calculate for this particular problem. So, given this omega vector is 0, 0 and omega 3, so our L is going to be, so M times. So, we have y square plus and z is z0 square, then we have xy and then we have minus xz and the diagonal, let us write down the diagonal components first, so this one is z0 square plus y square, and this is x square plus y square, and we have this component by, this is, Note that, this is a symmetric matrix, it turned out to be a symmetric matrix.

So, We need only six elements, so this will be xz, this will be yz, this is yz0, 0, 0, omega 3 and that gives us, so this is a 3 cross 3, this is a 3 cross 1, so the resultant is a column vector, 3 rows and 1 column. So, what you get, first term gives nothing, second term gives nothing, third term gives D and There is a M, and every term will have a M. So the third term will give M omega 3 xz0. Similarly, this second term is going to be minus M omega 3 yz0, and the third term is going to be M omega 3 and x square plus r square, oh, i square from the figure. is r square, so it will be clear if you look from the top, so this is our, this point centre, this is our point mass and this is any point with x, so if you look at it from the top.

It is clear that x square plus y square is r square, so this is r square. So, let us check that we get the right answer. So we have found a relation between angular momentum and angular velocity as a matrix product for a single point mass. So, let us generalize the definition of angular momentum to a extended object. So first, let us consider a collection of point mass.

Angular momentum for extended objects

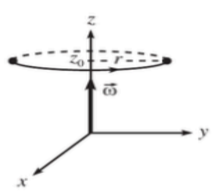
Collection of point mass $\mathbf{L} = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)$

$\sum_i \vec{r}_i \times \vec{p}_i$

There are n point mass, not a continuous mass distribution. In that case the angular momentum is straightforward, you compute this quantity r cross p. So, this is r, so if you take this here, M times, so omega times r is v, v of the particle i, vi, So this is vi and then this Mi times vi is pi, so this is essentially ri cross pi and then sum over all particles. So it is just a straightforward generalization from the definition we discussed before. Now let us take the example of taking two point mass in space.

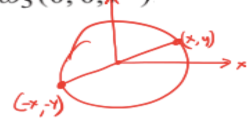
So in this case, again, It is the same problem, but now instead of a single point mass, we have two point masses. And these two point masses are diametrically opposite, given that they are travelling in the same circle about the same origin but are diametrically opposite. And again, the angular velocity of each of them is omega, which is same as before, and question is the same that calculates the moment of inertia I and angular momentum L. So, diametrically opposite means, let me explain that, so let us say if I have a top view and take the diameter of the circle, and then the masses are situated at the end of this diameter, Which means if this is the axis parallel to x axis, this is the axis parallel to y axis, and if this points coordinates are x and y, then the other points coordinate is minus x and minus y. So, let us calculate the moment of inertia matrix first.

Example 37: two point masses in space



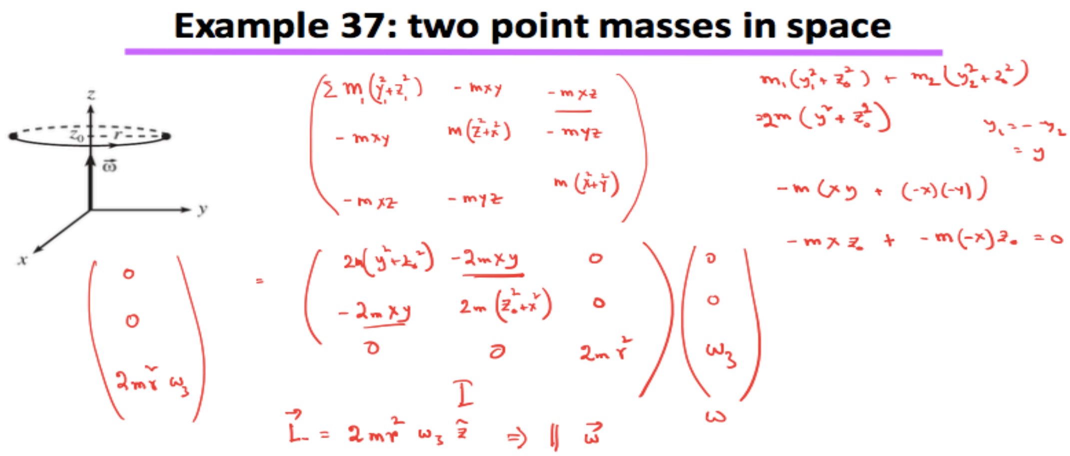
Two point masses in space, travel in the same circle, at diametrically opposite point. Given $\boldsymbol{\omega} = (0, 0, \omega_3)$
Calculate moment of inertia ***I*** and angular momentum **L.** Show that off-diagonal entries I_{xz}, I_{yz} are zero due to symmetry of mass distribution about z axis, but I_{xy} is not zero.

Ans: $\mathbf{L} = 2m\omega_3(0, 0, r^2)$



So we can, let us start with the expression that we got earlier and generalize this. So what we got earlier is the moment of inertia matrix is the following:. So we had M for a single; I am writing first for the single one point mass. So, now this was Mxy, this was Myz, this is by symmetry Mxy, this is minus M, sorry this is xz, this is yz, this is by symmetry Mxz, and this is minus Myz. Now we have two point mass, so we have to sum over.

So we have to write down that the first term will become M1y1 square plus z 0 square plus M2y2 square plus z 0 square. So, note that the z coordinate is same for both particles, and y 1 is minus y2, but square is again the same. So we can simply write it as M, and the masses are equal, so we can write it as A square plus z 0 square, and so y1 is equal to minus y2 equal to y, so this is just two times y square plus z 0 square. So then by symmetry, the other diagonal elements will be, and this will be x square plus r square for both of them.



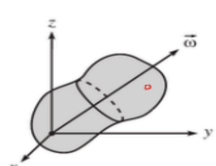
So, that will be r square. Now this term will become the minus Mxy and minus x minus y, so this will become minus 2 Mxy. But if I look at this term, This is minus Mxz0 and minus M minus xz0. So only x changes, z0 remains the same. So, this must be 0. And by the same argument, so the term containing only z and some other coordinate, all of them should be 0.

If this term is 0 by symmetry, this is 0. This is 0, and this will be 2Mxy. So this is going to be our, the moment of inertia matrix, and it shows that the term containing z are 0, but the term containing x y are non-zero. Now, if we multiply by the omega, what is our L matrix? Our L matrix: let us now notice this. So the first term is now 0, the first component. The second component is also 0, and the third component is 2Mr square omega 3.

Angular momentum for extended objects

Continuous objects

For a point mass *dm*, $\mathbf{r}\times\mathbf{p} = (dm)\mathbf{r}\times\mathbf{v} = (dm)\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})$



$$\mathbf{L} = \int \mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})\, dm$$

Angular momentum **L**, and angular velocity **ω** need *not be in the same direction*

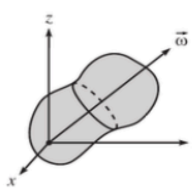
So notice a very interesting thing that in this example, the momentum, the angular momentum, is only in the same direction as the momentum. So it is possible that the angular momentum can be in the same direction or But it can also be in a different direction. So what is the difference between this example and the previous one? Notice that in this case, this mass distribution is symmetric about the z axis. It is not symmetric about the x and y axis; it is symmetric about the z axis. And that is why the x and y component turn out to be 0, as we saw mathematically, and that is why the angular momentum turns out to be in the same direction as the angular velocity in this case.

Now we can generalize the definition of angular momentum to a continuous object with continuous mass distribution. So instead of a point particle, we have now a point mass located at some position d M inside this body, and by applying the same concept. We just have to replace the sum in the earlier case by this integral over the mass distribution. But again, we remember that this resultant angular momentum can be in the same direction as the angular velocity, but they need not be in the same direction. So, this is the very sort of counterintuitive fact that sort of makes sometimes rotational motion more strange than they should be.

So keep that in mind. And now I show you the sort of the relation between angular momentum and angular velocity for an extended object in a matrix form. So here for this, this is a, so notation, shorthand notation for integral over the mass distribution. So this is the moment of inertia tensor. So, now we know what each component of this tensor means. So the diagonal elements of this tensor represents the distance of the spread of the mass distribution above the axis.

Angular momentum for extended objects

Continuous objects



$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \int (y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int (z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int (x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\vec{L} \equiv \mathbf{I}\vec{\omega}.$$

Tensor

$\int \Rightarrow \int dm$

So now all the things that we learned about moment of inertia tensor, we can immediately apply here. So we can choose, so we know that this is written in arbitrary choice of reference frame. This diagonal elements represents the mass and spread of the mass distribution about those x and y and z axis. And off-diagonal elements represent symmetry. Whether the mass distribution is symmetric, that is, the octagonal elements are 0 or non-symmetric.

So, this is clear now. And we have to just have one slight modification that this off-diagonal element now has a minus sign. So the products of inertia that We defined earlier did not have this minus sign. This is a minor sort of matter of convention. But we like to define this matrix. Some textbooks define it with a minus sign in the moment of inertia matrix.

We prefer to keep all the elements have same symbol without any minus sign and include the minus sign in the definition of the off-diagonal elements. And then, if you remember that if you have moment of inertia, so this is the tensor moment of inertia. So multiply by the omega, and the angular velocity will give you the angular momentum. So to summarize what we saw today, angular momentum is a vector, angular velocity is another vector.

But these two vectors are connected by the moment of inertia. So, moment of inertia, which is a matrix, means that angular velocity and angular momentum need not be in the same direction. Thank you.