Course Name: Newtonian Mechanics With Examples
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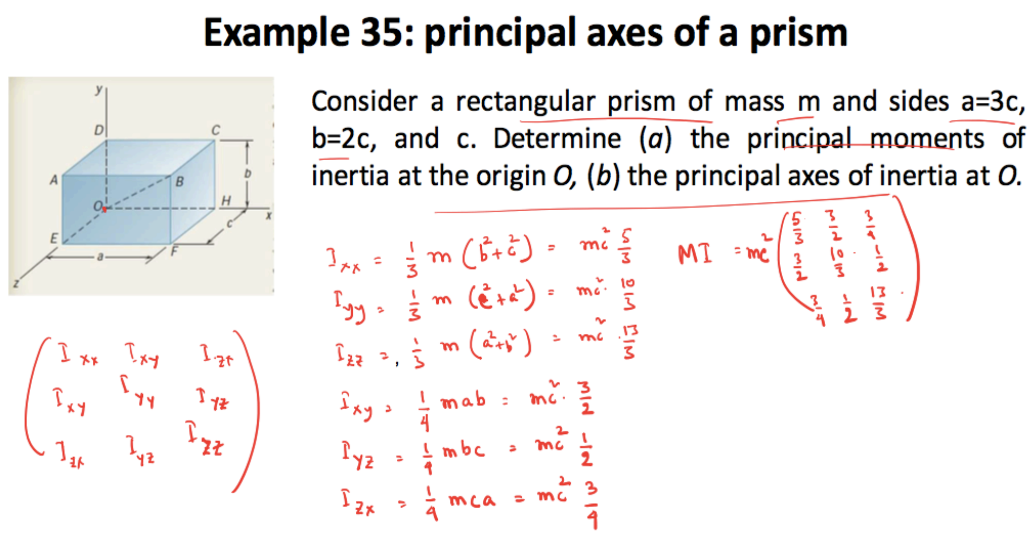
Week 08
Lecture – 40

In the last lecture, we discussed a very interesting concept called the principal axis of rotation. So, let's start today's lecture by considering an example. So, again, we take the same example of this rectangular prism of mass m with different sides A, B and C. Now, it is given that A is equal to 3C and B is equal to 2C. And the question is: that to determine the principal moments of inertia and the principal axis of inertia about the origin O. So, again, the same origin at the corner and the x, y, z axis are the sides of the prism.



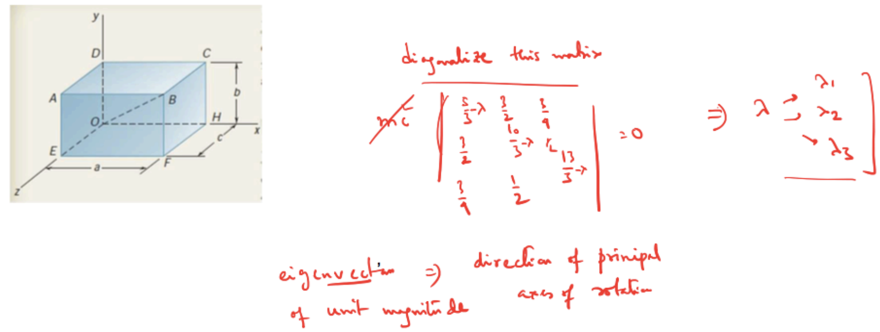
Now, given this system, we have already calculated. In one of the earlier example, we calculated the components of the principal axis, principal the moments of inertia. So, let us recall that we got the I x x to be 1 by 3 m B square plus C square and I y y to be 1 by 3 m A square, Sorry, C square plus A square. So, now, this B is equal to 2C.

So, this will be. So, we can take the m C square outside and this will be 5 by 3. And I y y, similarly, A is equal to 3C. So, this will be m C square times, so 10 by 3 and I z z was 1 by 3 m A square plus B square. So, this will be m C square times, so 9 plus 4, 13 by 3.

So, these were the components. And the products of inertia were I x y was 1 by 4 m A B, which is, So, This is 3 and this is 2, so this is m C square times 3 by 2. I y z was 1 by 4 m B C. So, this is m C square times, so this is 2 by 4, which is half and I z x is 1 by 4 m C A. So, this is m C square times, so A is 3C, so 3 by 4.

So, then we can write down a moment of inertia tensor matrix, which is the first diagonal terms are I x x, I y y, I z z. So, this is, I am going to take this m, so remember, I am going to write it here.

Example 35: principal axes of a prism



So, I am going to represent it as a matrix; then we have x y terms and this is a symmetric matrix. So this is I y x, which is symmetrical, the same as I x x y; this is I y z terms. So by symmetry, this is I z y is I y z and this term is I z x, which is same as I x z So, now for all the terms m C square part is common, so the first part is 5 by 3, The second y y term is 10 by 3, third term is 13 by 3, x y term is 3 by 2, y z term is half, so and z x term is 3 by 4, so then by this is the 3 by 2, this is half and this is 3 by 4.

So, This is the matrix moment of inertia in matrix form that we have determined in this particular choice of reference frame. So, the question is to sort of diagonalize this matrix, so diagonalize this matrix. So, I am going to write the, so first thing is will be 5 by 3, 10 by 3, 13 by 3, Then, we have 3 by 2 and 3 by 4, this is 3 by 2, 3 by 4, this is half, and this is half minus minus lambda equal to 0, so this goes up. So, this gives us the Eigen value equation, the lambda and there will be 3 possible choices of lambda: 3 Eigen values. Because in 3 dimension we have 3 possible axis of rotation, which corresponds to 3 possible moments of inertia above 3 possible axis, so these are the 3 Eigen values.

And once we get the Eigen values, step 2 is to get the Eigen vectors, so in general this Eigen vectors will give us the direction, so if I take the Eigen vector of unit magnitude. That will give us the direction of the principle axis of rotation. Now, this diagonalization. Now a days, there are standard ways that you learn in your mathematical methods course or In a computational analysis course, I ask you to complete this calculation and find out the Eigen values and Eigen vectors. And the Eigen vectors of this, once you find the diagonalized diagonal matrix that will represent the Eigen Eigen directions and those will give you the the direction of the principle axis.

Now, with this background, we have some background for an extended object. We have computed 2 properties that, even though we have an extended object. If I still want to represent by some representative point, that will be the center of mass and How to locate the center of mass? This is something we have discussed. And then we said that when we talk about rotation of an extended object, we need another quantity. Which sort of represents the extent of mass distribution about the axis of rotation and That is the moment of inertia.

With that background, now we are ready to do the physics part. Which is that to describe the rotational motion of a rigid body. Now here is a picture in which I showed with an example, some possible types of motion. Which are allowed for a rigid body. So, for example, we have a translation along a straight line in which the magnitude of the direction of the motion is constant.

And this motion is possible both by an extended object as well as a point mass. Second object. The second example is a translation along a curvy path. For example, it could be a circle. In this case, the direction of the motion can also change.

For example, in the circle is continuously changing. This is also a motion. That is allowed for a point mass as well as an extended object. For example, in the third translation, the third type is a rotation and in the simplest case. We have a planar rotation or a fixed-axis rotation.

So, we have an axis of rotation. Which is fixed and the body, and let us assume that the axis of rotation is passing through the object, and then this is rotating about itself. So, this is an example shown here like a compound pendulum. So, this is something that point mass cannot do because a point mass, a point cannot rotate about itself. So, this is what the difference between an extended object and a point mass.

And then we can have a more complicated motion. Which is a combination of translation and fixed-axis rotation. So, this is called a general plane motion. This is a plane because if you take one point and draw its trajectory with time. You will see that it moves in a plane.

So, here is an example of a mechanism which is called a crankshaft mechanism. So, here, some part is rotating, and the rest other part is translating. So, we took this example when we discussed the principle of virtual work and constraint motion. So, but this is also an example where you have a combination of translation and rotation. Now, translation versus rotation.

What are the similarities and differences? Now in mechanics, you can essentially ask two questions about the motion. The first question is: how do we describe the change of position with time? And for that, In the case of translation, we use the displacement. we can monitor the displacement of an object and then how does the rate of change of displacement, which is the velocity, and the rate of change of velocity, which is the acceleration. Now for the rotation, instead of displacement, we have to monitor the displacement of the angular change. The change in angle and we monitor the rate of change of the angle.

Which is given by the angular velocity and similarly angular acceleration. The second question is: what is the cause of the change in position? Why is the object changing position? How do we explain that? And this requires the laws of motion and here is one important quantity. That every object has something called inertia, which in the case of translation, the important part, it enters as a mass of the object. In the case of rotation, it enters as a moment of inertia in the laws of motion and In this case, we know the momentum principle, that says that it is the cause that is changing the position is the force and the effect of force or the external force on our system. The effect of force is to change momentum.

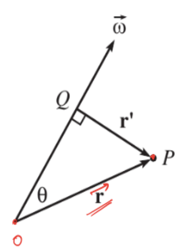
We can have an analogous quantity here. So, in the case of rotation, the quantity analogous to momentum is angular momentum, and the cause of the change in angular momentum is torque. So, we are going to discuss this: the relation between these quantities in some more detail. However, before that, I want to sort of put it as a side remark, some example of motion that we are not going to consider in this course because, in this course, we are assuming that our extended object is a rigid body, but if in real life, we often see objects that are extended but not so rigid. Here is a very famous example of a falling cat.

So, You can go to YouTube and find several videos where they show the experiment that you take a cat, hold it upside down in this particular position and then release it from some height, it most of the time or almost always lands on its feet. So, it sort of turns in the air and lands on feet. So, the puzzle in this case is that if you release the cat from rest, then its angular momentum is not rotating. So, if the cat was a perfectly rigid body, then it is impossible to turn itself, but the cat is actually not a perfectly rigid body. It is an elastic body, and different parts can deform with respect to each other.

So, this particular phenomena of turning of the cat is called zero angular momentum turn and instead of going into more details, I refer to you some link on the YouTube. Where you can see some real actual experiment with falling cats and explanation of the interesting physics behind it. But in this course, we are not going to take those kinds of objects. We are going to take extended objects which are perfectly rigid. So, let us our plan is to first recap some basic concepts that we sort of described here.

Relation between linear velocity and angular velocity

Theorem: Given an object rotating with angular velocity $\boldsymbol{\omega}$, the velocity of a point at position $\mathbf{r}$ is given by



$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

$$|v| = |\omega|\,|r| \sin\theta = \omega r$$

So, that, we are going to critically review these concepts and their relation in between them. So, we start with the angular velocity. So, first, let us say that we are considering a simple situation about rotation about a fixed point, the pivot point. Now, the first thing is that if our system is a point which is one-dimensional, Then, in 1D, there is only translation and no rotation. So, in a perfectly, if we lived in a one-dimensional world.

There would have been nothing like a called a rotation, only translation. Now Let us imagine what happens in two dimensions like this disc. So, the disc is rotation and In two dimensions, all rotations take place in that plane. In a two-dimensional plane, some object, some mass is rotating, like an extended object. Which is living on the plane or a point.

Which is living on the plane. So, all the rotations is happening in a plane, like for example, a circle or in some complicated shape, then it is possible just to have a level. So, The axis of rotation is always perpendicular to the direction of the plane. Hence, You can describe the rotation just by its magnitude, or speed. The direction of the rotation is always fixed and is always perpendicular to that plane.

Angular velocity is defined only for a given reference frame

Theorem Let coordinate systems S_1, S_2, and S_3 have a common origin.

Let S_1 rotate with angular velocity $\boldsymbol{\omega}_{1,2}$ with respect to S_2.

Let S_2 rotate with angular velocity $\boldsymbol{\omega}_{2,3}$ with respect to S_3.

Then S_1 rotates (instantaneously) with angular velocity

$$\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$$ with respect to S_3

There is no other possible choice. So, you can pretend that the angular velocity is just a magnitude. So, how to describe it? So, first, we identify the axis of rotation such as the one shown here. Then the vector, consider a vector, which whose magnitude is the angular speed omega and its direction is along the axis of rotation. So, remember that the axis of rotation, by definition, is fixed.

The direction is fixed. Now there are two possible directions: the up direction and the down direction and you can choose which direction by applying the right-hand rule convention that you curl your right-hand finger in the direction of the rotation or the spin. Then the thumb points, so the thumb points along the axis of rotation and its direction is the direction of omega. So, in this case, the direction is kind of fixed, and the only thing that matters is the magnitude. Now, in our actual real world, the 3D world, there are three possible independent planes.

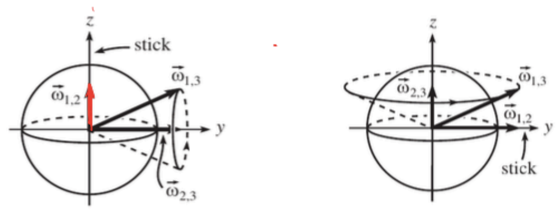
So, the crucial word is independent. So, in order to describe the rotation, We have to take not a single plane but three possible independent planes. So, for each of these planes, the orientation can be specified by the direction which is normal to the plane. And as we know, in 3D, there are three possible independent, not mutually perpendicular direction. Hence, the rotation takes place in three independent planes. So, you have to specify the angular speed in each of those planes.

So, there are three planes. So, there are three possible angular speeds. So, that means we can sort of guess that in 3D, the angular velocity, describing the angular velocity, we need three components in the most general case. So, that sort of tells us that angular velocity is a vector. Now, to be precise, the angular velocity is technically, so it behaves in the same way as a vector under coordinate transformation, but it behaves slightly different way under a reflection. So, under a rotation, it behaves same way as the position vector, but under reflection, it behaves in a slightly different way.

So, technically speaking, this is called a pseudo vector, not a vector, but pseudo; both pseudo vectors and vectors, they have magnitude and direction. So, for our purpose there is no distinction between a pseudo vector and a vector. Now, to define formally the angular velocity, we say that given an object which is rotating with an angular velocity omega, now we know that the speed of the angular speed is the magnitude and the direction given by the right-hand thumb rule along the axis of rotation is its direction. Then, it is related to the linear velocity, the velocity of the particle of a point particle, the velocity of a point at a position r, so let us say this is our origin and this is a point which is rotating with some in particular in 3D. Then, this direction of this point, the position vector of this point is r, vector r, so This is a bolt face, which means this is a vector quantity.

Example: rotating sphere

A sphere rotates with angular speed ω_3 around a stick that initially points in the $\hat{\mathbf{z}}$ direction. You grab the stick and rote it around the $\hat{\mathbf{y}}$ axis with angular speed ω_2. What is the angular velocity of the sphere, with respect to the lab frame, as time goes by?



1 → sphere
2 → stick
3 → lab frame

$$\boldsymbol{\omega}_{1,2} = \omega_3\hat{\mathbf{z}},$$
$$\boldsymbol{\omega}_{2,3} = \omega_2\hat{\mathbf{y}}$$

$$\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} = \omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$$

Then, its linear velocity, the actual velocity v. Which is a rate of change of displacement and is related to the position vector, the location by this particular expression. You are familiar with this, perhaps you are familiar with this expression in the magnitude, which is a two dimensional version. Whereas we only need to consider the magnitude, in full 3D we need to describe it as a full vectors using the full vector notation. So, the linear velocity is the cross product between the angular velocity and the position vector, so that defines the angular velocity. Now, here is one important point to note: angular velocity is defined for a given reference frame.

So you have to specify the reference frame when you specify the angular velocity. So, for example, suppose, so this is also true for linear velocity. You have to specify when you say a car is moving or with a speed 60 kilometer per hour. So, you are basically saying that the car is moving with respect to a reference frame of the road. If you are sitting inside the car, the car is actually not moving with respect to you at all.

This is also true for angular velocity. For example, if you have three coordinate systems or three reference frames S1, S2, and S3. Let us say that all of them has the same common origin. Then S1 is rotating with an angular velocity omega 1, 2 with respect to S2 and S2 in turn is rotating with an angular velocity of omega 2, 3 with respect to S3. Then S1 rotates with respect to S3 with an angular velocity.

Which is the sum of omega 1, 2 plus omega 2, 3. So, this is like addition of velocities, so the angular velocity is always relative to a particular given reference frame. So, let us try to take an example to understand this. Here is a sphere which rotates. So, there is a sphere, this is a sphere; and there is a stick; this is a stick. Which is initially pointing in the z direction as shown in the left figure and this sphere is rotating about this particular stick with an angular speed of w3.

Now, what you do is that. Now, you grab the stick and then you start rotating the stick about the, rotate it around the y axis with an angular speed of w2. So, this is the stick, which is now rotating about the y axis, which has a speed of w2. Then the question is what is the angular velocity of the sphere with respect to the left frame. So, in our notation that we used earlier, So, this w3 along z hat, so w3 along z hat, this is the angular speed of the, so there are three frames: the frame of the sphere, the frame of the stick and the left frame in which we are observing. So, w3 is the angular velocity of the sphere with respect to the left frame.

So, this is the angular velocity of the sphere. So, the angular velocity of the sphere with respect to the stick and omega 2 which is shown here, this is omega 2, so the stick is now rotating about x axis, about the y axis, so that direction is along the y axis, which is the axis of rotation. So, the angular velocity of the stick is, so you take, so this is. Let us say this is a z axis and this is y axis, so you are starting to rotate the stick about this direction by hand and the sphere is moving around the stick in this direction. So, the angular velocity of the stick with respect to the left frame is omega 2 y hat. Then, by what we learnt so far, so far about the addition rule of the vector, the relative velocity, so if you want to know the angular velocity of the sphere in the left frame, this is now doing a complicated motion.

It has two rotations: one is the rotation around the stick, and plus the rotation of the stick around the y axis. So, the net angular velocity of the stick is now omega 3, Which is the sum over omega, the sum of the angular velocity of the stick plus the angular velocity of the sphere with respect to the stick. So, now it has two component, so the direction of the angular velocity of the sphere is now along this particular direction. So, it is now doing some sort of a complicated motion.

Now, let us come for the sake of completeness. Let us point out put a brief remark about how to derive this relation? So, I am not going to derive it, give you the proof, but I just want to point out the basic logic by which we can deduce this particular relation So, we can use, start with this notation, rule that v is equal to omega cross r and then for the v, the linear velocity, you apply the relation for the linear velocity, they will just add and then you apply this relation, and you can deduce this particular relation. So, just to summarize today, we reviewed basic concept for describing rotational motion. Which is the angular velocity and the angular velocity in 3D in the most general case is a vector with three components, and its direction is along the axis of rotation. In the next lecture, we shall review the relation between angular velocity and angular momentum and It will be clear why the moment of inertia is not a simple scalar quantity; it has to be a matrix. Thank you.