Newtonian Mechanics With Examples

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Week -01

Lecture - 04

Let us continue our discussion with scalars, vectors, and tensors. So, in the previous lecture, we considered the elementary vector operations, the addition and then we discussed the dot product. So here, we will continue with the second kind of product between two vectors, which is the cross-product. So, if you take two vectors, now we are going to make an object that has both magnitude and direction out of these two vectors. So, this is a vector out of two vectors, this is a cross-product. So, again, we take two vectors, so A and B and these written in terms of the components.

$$\vec{A} = A_x\,\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\,\hat{x} + B_y\hat{y} + B_z\hat{z}$$
$$A = |\vec{A}|$$
$$B = |\vec{B}|$$

So, this is denoted by the symbol cross or X between A and B, so A cross B, and then you have a bit complicated-looking rule that is difficult to memorize. But this is not random or haphazard. So, this is a particular specific combination of the components. There is a simple way to remember this formula, which is to write down this formula as a determinant. So, in the first row, you write down the unit vectors, second row, you write down the components of the vector A. A is the first vector. As we shall discuss very soon, you have to keep track of the order.

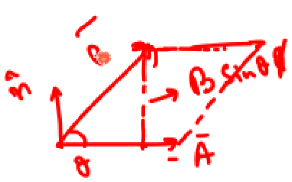
$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The vector that comes first goes to the second row, and the second vector goes in the last row. And then, if you pretend that this is like a determinant of a matrix and you apply the standard formula for the determinant of the matrix, you get back this formula. But remember, this is just a kind of pretension because, in a matrix, all elements of a matrix are numbers, whereas these are not just simple numbers, they are vectors. They are not scalar quantities. So, this is just a kind of mathematical convenience, but this is not a determinant of any matrix.

Now, there is a second way, just like in dot product, we can define the dot product, in terms of the magnitude and angle between vectors. The cross-product is defined as the magnitude of A, the magnitude of B, which is the second vector. Now, the sign of the angle between them and then this represents the magnitude, and then there is a vector n, this is a unit vector n, which represents the direction of A cross B. So, what is n? So, n means normal which means perpendicular.

$$\vec{A} \times \vec{B} = A\,B\,\sin\theta\,\vec{n} \quad ; \ \theta \text{ is angle between } \vec{A} \text{ and } \vec{B}$$

So, n is a direction that is perpendicular to both A and B. Again, when we see this definition or some rule, some formula, you have to ask again, does it make sense? and again, the first strategy that we learned is to draw sketches. So, let us say that this is my vector A and this is my vector B and this is the angle between A and B. So, again if I drop a perpendicular, then what do I get? What I get is that I drop a perpendicular from the tip of B on A. Then, this represents the height of this triangle, that is B sinθ.



$A(B\sin\theta)$ = area of this parallelogram

That means that if I complete this parallelogram by drawing the parallel of the vector A and the parallel of the vector B, then AB sinθ represents the area of this parallelogram. So, in general, it is a parallelogram. If A and B are perpendicular to each other, then this will be a rectangle. So, I will leave it as an exercise for you to verify this statement, for this case, when A and B are perpendicular to each other. Okay, so this represents the area or the magnitude.

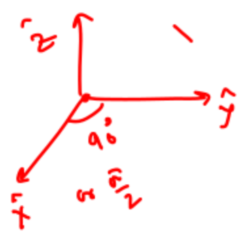
So, now you know that if I give you two different arrows, then, like two fingers, this defines a plane. So, if I give you two independent arrows, then they define a plane. So, the rule says that the direction of the cross-product is perpendicular to both A and B, which, if you think, it means that the direction of the cross-product n hat represents the direction, which is perpendicular to the plane. So, this gives you the direction of n hat. Now, if you really try to do, take your finger, or two pens, or pencils, and try to make sense that which is the direction, you can easily see that the direction is not unique.

There are two directions possible. Suppose this is A, this is B, then the direction which goes from upward and the direction which goes vertically downward, both are perpendicular to this plane. But a single vector cannot have two directions at the same time. So, in that sense, if you just simply say, the direction is perpendicular to both A and B, it is little ambiguous. So, it turns out that we require a convention to pick up, to select the direction of the cross-product between A and B.

So, the convention is that you apply a right-hand thumb rule. So, if you start from A, if this is your A and this is your B then you apply your right hand, not the left hand. So, you start with a right hand and then move your hand in the direction of B then the direction of your right-hand thumb gives you the direction of A cross B. So, it is a very interesting point that to define the direction of the cross-product, you need to apply your right hand. And if you apply your left hand, you will get the wrong direction.

So, you need some convention to define this direction. The quantities for which you need this extra convention to define the vectors/directions are special categories of quantities, and these are called pseudo vectors. So, we will not go into the difference between the true vectors and pseudo vectors. We will simply mention examples that will be relevant to our course. These are the angular velocity, angular momentum, torque, etc.

Now again, we try to unwrap the formula that we just introduced, namely this for cross-product. So, let us say that we take some special cases. So, let us say that we calculate the cross product between x hat. So again, we take our Cartesian component system.



So, we want to know what is x hat cross x hat. If I apply this definition, a cross product of a unit vector with itself. So first, I need the magnitude of first vector which is 1, the magnitude of the second vector, which is 1, and sine of the angle. So, the angle θ of a vector with itself is 0, sine of 0 is 0. So, the cross product of a vector with itself is 0. So, this is a null vector essentially.

$$|\hat{x}\times\hat{y}| = [1.1.\sin(\tfrac{\pi}{2}) = 1]$$

$$\hat{x}\times\hat{x} = 1.1\sin 0$$
$$= 0 \text{ (null vector)} \qquad \hat{x}\times\hat{y} = \hat{z}$$

$$\hat{y}\times\hat{z} = \hat{x}$$
$$\hat{z}\times\hat{x} = \hat{y}$$

So, this is different from what we see from the dot product. Now, let us take two different unit vectors. Let's say x hat and y hat. So, x hat cross y hat. What does it give? The magnitude is 1x hat and y hat is 1. And the angle between x and y is 90 degrees or π/2. So, this is sine π/2, which is 1, so the magnitude is 1. Now, what about the direction? If you apply your right-hand thumb rule, you start from x-axis, and move toward y-axis, then your right-hand thumb will point to the z direction. So, the cross between x and y is the unit vector along the z-axis which is z-hat. You can verify that if you take the cross between y and z is x hat and the cross between z and x is y hat.

That is why I have drawn this coordinate system, called a right-handed coordinate system. So, let us take an example to understand this definition.

Given $\vec{P} = 2\hat{x} - \hat{z}$, $\vec{Q} = 2\hat{x} - \hat{y} + 2\hat{z}$, and $\vec{R} = 2\hat{x} - 3\hat{y} + \hat{z}$, find

(a) the component of $\vec{P}$ along $\vec{Q}$

(b) a unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$.

Let's do it one by one. The component of P along Q, here you encounter a question. So, what kind of product we are talking about, dot product or cross product? And here you can easily feel, if you have a good picture of the product between two vectors in your head, then it is easier to attack this problem. So, remember the physical pictorial meaning, so the component of P along Q, it represents a projection of P along Q. So, we are talking about the dot product. So, we want to calculate the dot product between P and Q.

$$\vec{P}\cdot\vec{Q} = P.Q.\cos\theta$$

So, this will give me PQcos θ. So, the projection of P along Q is Pcos θ. Now I want to go one step further, and I want to define a vector, which is in the same direction as Q, and whose magnitude gives you the component of P along Q. The component of P along Q will be Pcos θ, Now I want to draw a vector, whose magnitude is P cosθ, and whose direction is a unit vector along Q.

So, how to calculate this quantity? So, note that I can write this as, now I apply the definition of dot product and unit vector. So, I can write this as,

$$= \frac{QP\cos\theta}{Q}\hat{Q} = \frac{\boxed{QP\cos\theta}}{Q}\cdot\frac{\vec{Q}}{Q}$$

$$= \frac{\vec{P}\cdot\vec{Q}}{Q^2}\vec{Q}$$

$$\bar{P}\cdot\bar{Q} = 2.2 + 0.(-1) + (-1).2 = 2$$
$$Q^2 = 2^2 + (-1)^2 + 2^2 = 3^2 = 9$$
$$\longrightarrow \frac{2}{9}\left(2\hat{x} - \hat{y} + 2\hat{z}\right)$$

This represents a vector whose magnitude is the same as the projection of P, along Q, and the direction is parallel to Q. So, if you calculate the magnitude of this vector, that will give you this projection of P along Q, the P cos θ.

The next problem is, find

(b) a unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$.

So, this is a straightforward application of the cross-product. So, we need to calculate Q cross R and magnitude of this vector.

$$\bar{Q}\times\bar{R} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \hat{x}(5) + \hat{y}(2) + \hat{z}(-4)$$

$$|\vec{Q}\times\vec{R}| = \sqrt{5^2 + 2^2 + (-4)^2}$$
$$= \sqrt{45}$$

$$\hat{n} = \frac{5}{\sqrt{45}}\hat{x} + \frac{2}{\sqrt{45}}\hat{y} + \frac{-4}{\sqrt{45}}\hat{z}$$

So, this n hat which represents a unit vector, which is perpendicular to both Q and R. Now note that you can also define the vector minus n is also a unit vector, magnitude is the same and it is also perpendicular to both Q and R.

Ans: $\pm\hat{n}$

So, the answer is ± n hat. So, two answers are possible. So, this is important. So, putting this ± is important.

Now I want to point out two crucial differences between dot and cross products. So, the first difference is that

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

If you apply the rule that defines the dot product, you can easily verify that. Now you apply the formula for the cross-product, and you will see that,

$$\begin{aligned} \vec{A} \times \vec{B} &\neq \vec{B} \times \vec{A} \\ &= -\vec{B} \times \vec{A} \end{aligned}$$

So, this is a crucial difference between a dot and a cross product. So, the cross-product behaves like a real number. So, that is why it is called a scalar product.

But the A cross B, is not a multiplication of two numbers, it is a multiplication of two vectors. You can easily see that by applying your thumb, right-hand thumb rule, so if you go from A cross B, you are going from A to B, and the direction is upward. Whereas, if you go from B to A, then your direction is downward.

So, the magnitude remains the same, but the direction reverses. So, A cross B is not equal to B cross A. So, in that sense, you have to take the direction into consideration.

The next one is as we explain or verify for the case of unit vectors, if you take the product, a dot product of a vector with itself, you get the magnitude square.

$$\vec{A} \cdot \vec{A} = A^2 \longrightarrow |\cos\theta| \text{ is maximum} = 1 \text{ in this case}$$

$$\vec{A} \times \vec{A} = 0 \ \text{(null vector)} \longrightarrow |\sin\theta| \text{ is minimum} = 0 \text{ in this case}$$

Whereas if you take the cross product of a vector with itself, you get the null vector, 0. Because in the cross-product, what goes into the formula is the sine of the angle. And when θ is 0, the sine θ, magnitude of sine θ is minimum, which is 0. So, this is another difference between a dot product and a cross product.

So, don't get confused between these two products. Pay attention to the notation clearly, and remember that dot product gives you a number. Whereas, the cross product gives you a vector, which has both magnitude and direction. So, to be really technical, I should mention it as a pseudo vector. But for our purpose, unless under very special circumstances, which we will not require for this course mostly, so there is no difference between the vectors and the pseudo vectors.

Now, I am going to cover two more products. These are called a product between three vectors instead of two. So, how can you combine three vectors? And so, what will be the result? Will it be a number or scalar or will it be a vector? So, let us take three vectors A, B and C. Now you can see that if I want to make a dot product, I need two vectors. So, let us try to simplify three vectors down to two vectors. So, how can I combine two vectors and make a vector? The answer, as you know, now is to make a cross-product.

Let's say I take two vectors B and C, and then B cross C is another vector. So, it is a vector. And then A is another vector. I can take a dot product between this vector A and this vector B cross C.

$\bar{A}, \bar{B}, \bar{C}$

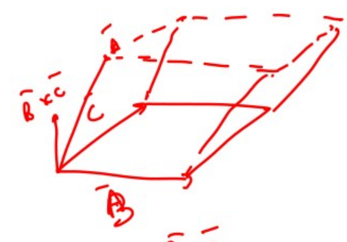
$\bar{B}\times\bar{C} \rightarrow$ vector

$\bar{A}\cdot(\bar{B}\times\vec{C})$ = number./scalar

✓ Scalar triple product

So, this is called A dot B cross C. This is a number because the final operation is a dot product or a scalar; as we can easily show, if you do a coordinate transformation such as rotation, this number does not change from one frame to another frame, from the blue frame to the red frame. So, this is a scalar or a number. So, this is called a scalar triple product. Since this is a product of three vectors, this is called a triple product. This is a number or a scalar, so this is called a scalar triple product.

So, we can give a sort of simple geometrical meaning to this quantity. So, I am not going to prove it. I am just stating it. So, if you have this vector A and another vector B, and this is a vector C, then you know that B and C together, they define a plane. And in this plane, if you complete the parallelogram, then the area of this parallelogram is B cross C and then the B cross C is perpendicular. So, the direction is perpendicular following the right-handed thumb rule.



$\bar{B}\times\bar{C}$

parallelopiped

volume = $\bar{A}\cdot(\bar{B}\times\bar{C})$

So, this is the direction of B cross C. Now, if you take another vector A, then the projection of this B cross C along A gives you the A dot B cross C. And one can show that if you have these three vectors, you can draw a three-dimensional, kind of deformed cube, which is called parallelepiped. If all of them are perpendicular to each other, this will be like a block, A block like a brick. So, this is like a distorted brick.

The volume of this distorted brick is given by this scalar triple product. So, this is something I am stating without proving, but this is a kind of a use or practical application of the scalar triple product.

So, the next question is, can I make a vector out of three vectors? The answer is yes. How? Again, you take these three vectors A, B, and C. Now if I take this pair, then I can make a cross product out of them. And then if I take a cross-product of A, previous case I took a dot product between A and the cross product of B and C. But you can also take a cross-product. And this is called a vector triple product because the resultant is a vector because it is a cross product. So, this is a double cross product.

$$\bar{A}, \underline{\bar{B}, \bar{C}}$$

$$\downarrow$$

$$\bar{A} \times (\bar{B} \times \bar{C}) = \text{vector triple product}$$

$$\bar{A} \times (\bar{B} \times \bar{C}) = \vec{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

BAC- CAB rule

This is usually known as BAC-CAB. This is a kind of useful mnemonic that helps you to remember the formula.

Now, is order important here? If I take the second cross-product first and then the first, first says if I take the first cross-product first and then take the second, are they the same? Let us check that. So, if I apply this BAC minus CAB rule, this basically gives me the formula. The rule basically says to take the middle vector, so write the middle vector first, and then take the dot product of the remaining two vectors, then take the last vector, and then take the dot product of the remaining two vectors. So now, if I want to apply it here, first, I have to write it in this particular form.

$$\bar{A} \times (\bar{B} \times \bar{C}) \overset{?}{\neq} (\bar{A} \times \bar{B}) \times C$$

order of crossproduct is crucial.

$$\boxed{\bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})}$$

$$(\bar{A} \times \bar{B}) \times \bar{C} = -\bar{C} \times (\bar{A} \times \bar{B})$$

$$= -\left[\bar{A}(\bar{C}\cdot\bar{B}) - \bar{B}(\bar{C}\cdot\bar{A})\right]$$

$$= \boxed{\bar{B}(\bar{C}\cdot\bar{A}) - \bar{A}(\bar{C}\cdot\bar{B})}$$

$$\bar{C}\cdot A = \bar{A}\cdot\bar{C}$$

So now, if I compare this expression with the previous expression when I take the second vector, then I get one term the same, but the second term, this term, they are different. So that means these two are not the same. That means if you have two cross products, one after another, you must always give a bracket to denote which cross product is taken first and which cross product is taken next.

The order is important. So, the order of cross product is crucial. This concludes a very basic and quick review through examples of basic math, the scalars, and vectors that are required for this course, which is on Newtonian mechanics through examples. So, in the next week, we shall start with the physics or review of Newtonian mechanics. So, we shall see you again next week. Thank you.