Course Name: Newtonian Mechanics With Examples
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Week 08
Lecture - 39

This is the final week of this course on Newtonian mechanics with examples. Let us start with a very interesting concept called principal axis of rotation. So, remember that when we discussed the moment of inertia which is required to understand rotation, we said our goal is to go through to be able to solve two kinds of problems. And so far, in the previous week, we discussed the first type of problem, which is that given a mass distribution of an extended object and a set of axis of rotation, compute different components of the moment of inertia with respect to the x axis, y axis, and z axis. Which sort of give you that how much is the spread of the mass distribution about those axis of rotation. And the second is to compute the products of inertia, which we considered in the previous lecture and we saw that this quantity gives us the, so if this is 0, This means that the mass distribution is symmetric about that axis and if it is non-zero, then That means the mass distribution is asymmetric about that axis.

Rotation: Moment of Inertia of extended objects

Type of problems

Type 1: Given a mass distribution (an extended object) and a set of axes of rotation,

a) Compute I_{xx}, I_{yy}, I_{zz} (recap of definitions, examples in 2D and 3D)

b) Compute products of inertia I_{xy}, I_{yz}, I_{zx}

Type 2: Given a mass distribution (an extended object) find the principal axes of rotation.

We have taken, say considered several examples in the last week of like where the reference axis. So, the origin was taken at the centre of mass for some and the axis of rotation, where the symmetrical axis of symmetry of the object. We also generalized this to the moment of inertia and we now know that we do not have to take only the axis of rotation or only origin at the centre of mass. We can put the origin at arbitrary location and choose arbitrary axis of coordinate system and even then, we are now able to calculate the components of moment of inertia.

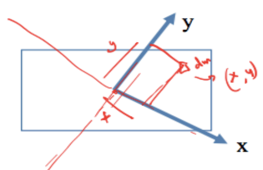
Today, our goal is to look at the second type of problem, which is given a mass distribution of an extended object, find the principal axis of rotation. So, let us start by understanding. What is the principal axis of an extended object? So, let us take a two-dimensional mass distribution because it is easy to visualize it in 2D. So, we take an example in 2D. So, this rectangle is some rectangular mass distribution.

Now, we are showing two different coordinate system, the black and the blue. Now, in the left figure, these two coordinate system are the orientations are parallel to each other. But the origins are different. So, you can translate the one coordinate system by some fixed along this particular direction, and then you can go to the next coordinate system. In the right-hand figure, the origins are same and then if you want to convert from the black-to-blue coordinate system or vice versa, you have to do a rotation.

So, in the left figure, you have to do a translation, and in the right figure, you have to do a rotation. Now clearly, if we look at the, let us say if I extend this figure little bit. It is clear that the black reference frame, black coordinate system, is a reference frame. In which, the mass distribution is symmetric about the origin. Whereas the blue reference frame In both figures, the mass distribution is asymmetric about origin in both cases.

So, here we pose a mathematical question. We ask what is the coordinate transformation rule? So, in specifically, let us say if we take the right-hand figure. How much is the rotation we have to give by an angle theta, so to go from the asymmetric coordinate system to a symmetric coordinate system. So, from to go from the blue to black or black to blue vice versa, So, we have to rotate by some angle theta and in this particular case. The axis of rotation is the z axis, which is perpendicular to the plane of the figure.

MI in reference frame of *asymmetric* mass distribution



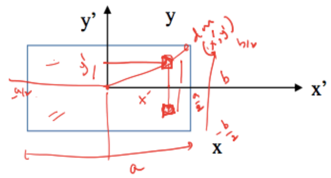
$$dm = \rho\, dx\, dy$$
$$= dA \text{ set } \rho = 1$$
$$I_{xx} = \int y^2\, dA \text{ MI about x axis}$$
$$I_{yy} = \int x^2\, dA \text{ MI about y axis}$$
$$I_{xy} = \int xy\, dA \text{ product of inertia}$$

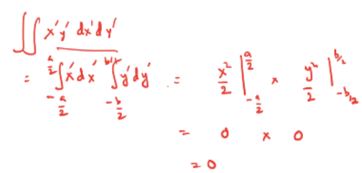
Moment of inertia in reference frame (x,y)

So, about the z axis we have to rotate by angle theta to go from one coordinate system to another coordinate system. The question is how much is the theta? Now, before we go to that problem, let us first write down, What are the components of the moment of inertia in the asymmetric coordinate system. So, if we take any point dm. Which is located at coordinate xy, so this is, so if I drop a perpendicular then this is y, this is x, and if I draw a perpendicular, this is y. So, then a mass element d m and rho is the density, and let us say rho equal to be 1, so omit the density.

Then, by now inertia about this particular x coordinate. So, the moment of inertia of this object with respect to this x-axis So, with respect to this axis, if I calculate the moment of inertia. That will be I x x and similarly, if I calculate this particular integral, x square d A. About the y axis, that will give me the moment of inertia about this particular y axis. And similarly, if we calculate the product xy times d A, that will give me the Ixy.

MI in reference frame of *symmetric* mass distribution





Moment of inertia in reference frame **(x',y')**

$$dx'\,dy' = dx\,dy = dA$$

$$I_{x'x'} = \int y'^2\,dA \text{ MI about x' axis}$$

$$I_{y'y'} = \int x'^2\,dA \text{ MI about y' axis}$$

$$I_{x'y'} = \int x'y'\,dA \text{ product of inertia}$$

So by figure, it is clear that this particular choice of the mass distribution is not symmetric about this particular choice of axis, so we expect by definition that this Ixy to be non-zero. Now, we can do the same thing for the black coordinate system, which is symmetry. In which, the mass distribution is symmetry. So, by symmetry, the middle point of this rectangular is the centre of mass and this is clearly passing through the centre of mass. Here, also, the axis is passing through the centre of mass, but the orientations are such that the mass distribution is asymmetric.

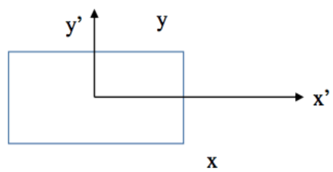
Here, the orientation of the axis is such that the mass distribution is symmetric. So, again, we can write down the components in this black coordinate system. So, I take the same element here, d A, so the mass is a scalar that does not change. If you change the coordinate system, Now this coordinate, but now the coordinates, the position vector is now different; now its component is at x prime and y prime, which is different from the components are different point is same, but the components are different. So, now this is your x prime and this much is y prime, but the area remains the same.

So, this is the area, which is the same. So, it is a property of the coordinate transformation that the area will remain the same. So, I heard of I am going to assume this result, Those of you who are interested, you will encounter. So, you have to calculate something called Jacobian of this coordinate transformation and you can easily show that the Jacobian is 1, so that the area will remain the same. Now, we apply the same definition, so if I take this x axis, then the moment of inertia about this axis will be given by Ix prime x prime, moment of inertia about this y axis given by Iy prime y prime and the product of inertia will be given by this particular quantity x prime y prime d A.
And since we see clearly that the mass distribution is symmetric about this axis, we expect that the product of inertia should be 0. Now in this case this is easy to see by doing this x by doing this x this integral. So, say that when I do this integral x prime y prime d A, now if I keep y fixed x fixed and then for this d m element, I take its reflection about x axis, that is x prime axis and this is the reflection, so it is the image of this element. So, the difference is that this has the same distance

from the x axis, but the sign is different. Then the contribution to this area and this area will have equal magnitude but opposite signs.

So, for both of them, x prime is the same, but y prime is equal and opposite in sign. So, if we add them, they will cancel each other. And similarly if I do it over this one quarter of this distribution, we can easily see that this one quarter of the distribution is going to cancel the lower one-quarter of this distribution. Hence and similarly, this part of the distribution is going to cancel this part of the distribution and hence we see that this Ix prime y prime is expected to be 0. So, you can actually work it out.

MI in reference frame of *symmetric* mass distribution



If mass distribution is symmetric about origin then product of inertia is zero in that reference frame.

Moment of inertia in reference frame **(x',y')**

$$dx'\,dy' = dx\,dy = dA$$

$$I_{x'x'} = \int y'^2\,dA \text{ MI about x' axis}$$

$$I_{yy'} = \int x'^2\,dA \text{ MI about y' axis}$$

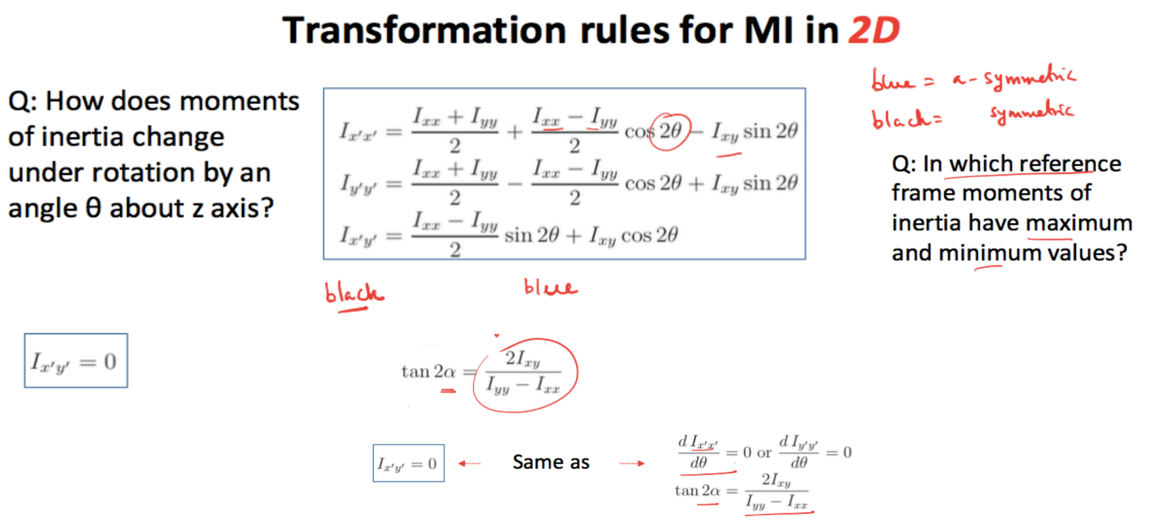
$$I_{x'y'} = \int x'y'\,dA \text{ product of inertia}$$

So, let me work it out. So, when I do x prime y prime dx dy, then y since this is a x and y in this particular case is a rectangular distribution, so x and y are independent. So, this is x prime y prime times y prime dy prime. Now let us say that this length is A and this side is B, so this is Ay2 and this side is B. So, x varies from minus Ay2 to plus Ay2 and y varies from minus By2 to plus By2.

So, the limits are minus By2 to plus By2. So, this gives me x square by 2 Ay2 to minus Ay2 times y square by 2 plus Ay2 to minus Ay2. So, this is x square By2 to By minus By2, and this is 0 into 0 which is 0. So, you now verify that if Ix prime y prime, the product of inertia is 0, then the mass distribution is symmetric. So, this is what we just checked: if mass distribution is symmetric about the origin, then the product of inertia is 0 in that reference frame.

So, this is the key idea to understand the principle axis of rotation. So, what we need is that here, so suppose in this case, It is easy to guess what this axis of symmetry is, and the choice of axis for above for which mass distribution is symmetric. But suppose we have some complicated shape of the object. so that it is not easy to guess what is the axis, the choice of axis about which the mass distribution will be symmetric. So, what can we do? We can start with an arbitrary axis, such as this blue axis, and calculate the moment of inertia.

Then we can ask this question: How do the different components of moment of inertia will change under rotation when I go from blue axis to black axis that is under rotation by an angle theta about the z axis. Now, I am going to give you the answer, and the answer is here. So, these are some transformation formulas. So, if we know the components of the moment of inertia Ixx Iyy and Ixy in the blue axis, so blue is asymmetric and black is symmetric. That is why, in the blue axis, blue is asymmetric and black is symmetric.



So, if we know the components in the blue axis, then we can calculate the components that will be in the black axis. By using this particular formula,. So, here is our first lesson that even if we have the same object, and origin but still the moment of inertia is not unique. So, this is something you should remember, depends on our choice of the axis or the reference frame. So, then we can sort of say that, so now we are going to use that particular principle, the particular idea that if, on the black axis, the mass distribution is symmetric, then the product of inertia must be 0.

So, we can use this condition to calculate the value of the theta. So, note that the right-hand side depends on the components in the black axis. Sorry, the blue axis as well as the angle of rotation theta. Then we can demand, so This is a very interesting part of this particular formula. So, I am going to show you and give you the result.

So, now if we demand that this product of inertia is 0. Which is the condition that the our black axis the mass distribution is symmetric. Then you demand that this be equal to 0, and you can easily show how to solve it for the value of alpha, theta and the particular solution is alpha. So, theta is equal to alpha, which is given by this particular relation between the components. This is very interestingly turns out that, We can also see the same condition if you ask a slightly different question.

So, the second question is, in which reference frame the moments of inertia have the maximum and minimum values, which means in which reference frame the Ixx and Iyy are maximum. So, what do you mean by that? So, as you can see, the moment of inertia depends on the choice of axis. Now if you take different values of theta, you can generate starting from your blue reference frame you can generate different possible infinite number of different possible choice of axis. And in each of those reference frames, you can thus compute the moment of inertia of the given same object. Now you can ask which frame will give me the highest value of the moments of inertia.

And it turns out that you not only get the highest value but you also get the lowest value in the same frame. So, the condition is that if you take this d Ix prime x prime and so this is a function of theta. Now, if you extremize this with respect to theta,. So, if you calculate the derivative with respect to theta and set it to 0, that is the condition of maximum or minimum, then you end up with the same value, which is the same value. So, same condition that gives you the Ix prime y prime to be 0 also gives you the same condition that gives you the Ixx prime and Iy prime to be maximum.

Moment of inertia w.r.t principal axes in 2D

$$I_{x'x'} = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'y'} = \frac{I_{xx}+I_{yy}}{2} - \frac{I_{xx}-I_{yy}}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_{xx}-I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{x'x'}^{max} = \frac{I_{xx}+I_{yy}}{2} + \frac{1}{2}\sqrt{(I_{yy}-I_{xx})^2 + 4I_{xy}^2)}$$

$$I_{x'x'}^{min} = \frac{I_{xx}+I_{yy}}{2} - \frac{1}{2}\sqrt{(I_{yy}-I_{xx})^2 + 4I_{xy}^2)}$$

Which means that the moment of inertia is maximum and minimum in that particular reference frame in which the mass distribution is symmetric about the origin. So, it looks like. There is a certain special given an object and some particular origin there is a certain special reference frame, which has two interesting property. The first property is that the mass distribution is symmetric about the axis in that reference frame. And the second property is the moments of inertia are have the largest and the lowest possible values.

So, this particular set of reference frame is called the principal axis of rotation is a special given an object is a property of the object. You can always define a particular reference frame in which the mass distribution is symmetric. So, no matter, An object can be very highly irregular, but it always has a particular choice of axis of symmetry. And here I write down the value. So, if you plug in this condition that this tan 2 alpha is using this condition and In this particular equation, you can derive this formula.

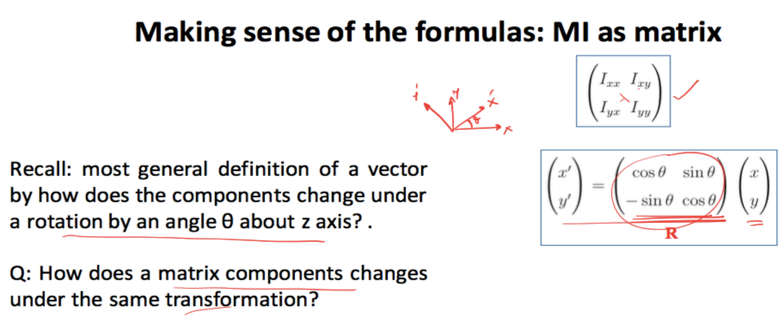
So, I put it as a take-home exercise to work out the algebra; it is just two, three lines of algebra. So, you can see that this is some form of a plus b and a minus b. So, This must be the maximum value. So, note that the second term is the same in both the equations, and the second term is always a positive quantity. So, this must be maximum and these must be the minimum values of the moment of inertia.

So, now the question is that this is fine. So, we appreciate that there is a given object. There is always some axis of symmetry fine and then there is if you want to calculate the moments of inertia about that particular axis, we have this particular axis's particular complicated set of formulas. And now another beautiful thing is that You no longer have to remember this particular complicated set of formulas. You have to remember just one rule, and this is the following: we are going to use the full power of vector and matrix algebra. So, you are going to represent the moment of inertia as a matrix.

So, in 2D we have going to write the different we have three different components Ixx Iyy and Ixy. So, We are going to arrange them in a 2 by 2 matrix in this particular order and the fourth element Iyx will be by definition it is so this is symmetrical. So, we have a 2 by 2 matrix in 2D. So, the point is that if we take an arbitrary reference frame, we do not know what the axis of symmetry is. We can allow to take an arbitrary reference frame, then in general the these two components are non-zero.

And if we are computing these components in the symmetric axis of about the axis of symmetry, then these two components are 0. Now here we recall what we learned in the first week of this course, that we discussed several different ways of defining vector and we introduced the most general way of defining vector, is that how it behaves under a coordinate transformation, such as under a rotation by an angle theta about z axis. And we show that the vectors are quantities, such as the position vector is that, if you take a, this matrix represents the rotation by an angle theta counterclockwise rotation by an angle theta about the z axis. So, we start from this particular some choice of axis and then this is our y prime, and this angle is theta. Now, we have to write the this geometrical operation we can represented by this particular 2 by 2 matrix.

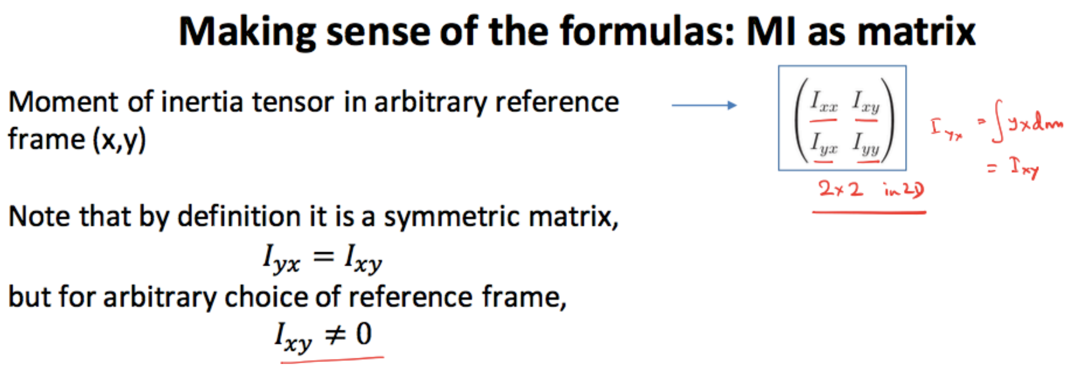
Then the question is So this is about the vectors, which can be represented by a column vector. Now, what about the 2 by 2 matrix? How does a component of a matrix change under the same transformation? So here I am just going to state the rule, and this rule is particularly easy to remember. So, this is the rule you take the matrix, which is your 2 by 2 matrix and then you multiply on the left by the same rotation matrix R, which is a function of theta and from the right-hand side, it transpose of the rotation. So, this is called the rotation operator. So, this is hits this matrix and rotate it coordinate system by an angle theta.



So, then You can easily so I will keep it again as an exercise for you to verify. You can check for yourself that if you take this particular matrix with the product of these 3 matrices. You get exactly the same formula as listed here. So, you do not have to remember that this complicated-looking formula, you just have to remember the following fact: The moment of inertia in 2D is a 2 by 2 matrix of this particular, where the elements are arranged diagonal elements are the moments of inertia above the axis and off diagonal elements are the products of inertia, and then in this part, this is the rotation operator. So, this is the rotation operator and then the transformation rule is multiply from the left hand side by R and multiply from the right hand side by transpose of R.

So, the objects in general of mathematical quantities that follow this particular transformation rule are called tensor of rank 2. So, that is why we discussed this particular way of defining scalar vectors and tensors in the first week so that we can now use them. So, this is something you should remember: that moment of inertia is just not a single number it is a matrix and a matrix is a tensor of rank 2 it is not just a simple quantity with a magnitude and directions much more complicated than that. So, the physical meaning will be clear in the coming lectures.

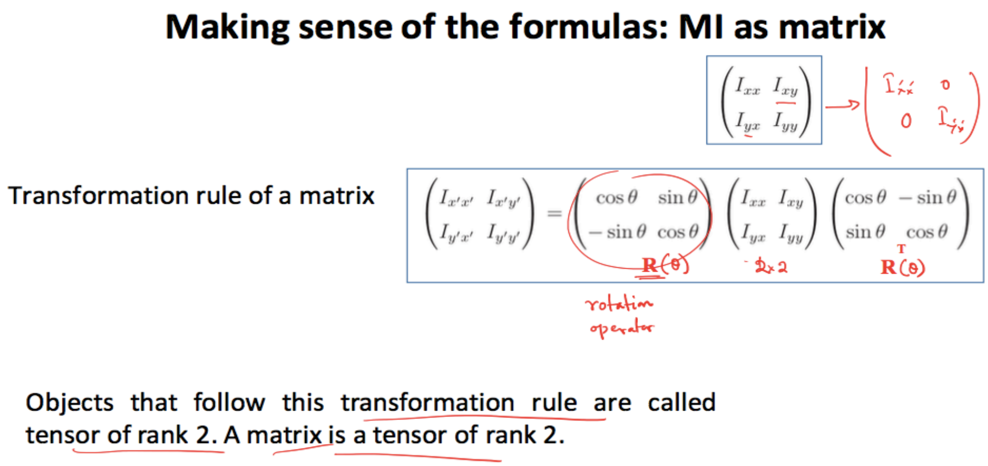
So, then, finding the principle axis of rotation. We can pose this problem as a matrix. So, we demand that the in the suppose We started this matrix, and we demand that we go do some transformation on this matrix. So that it become the off diagonal elements become 0. That means we are talking about matrix diagonalization.



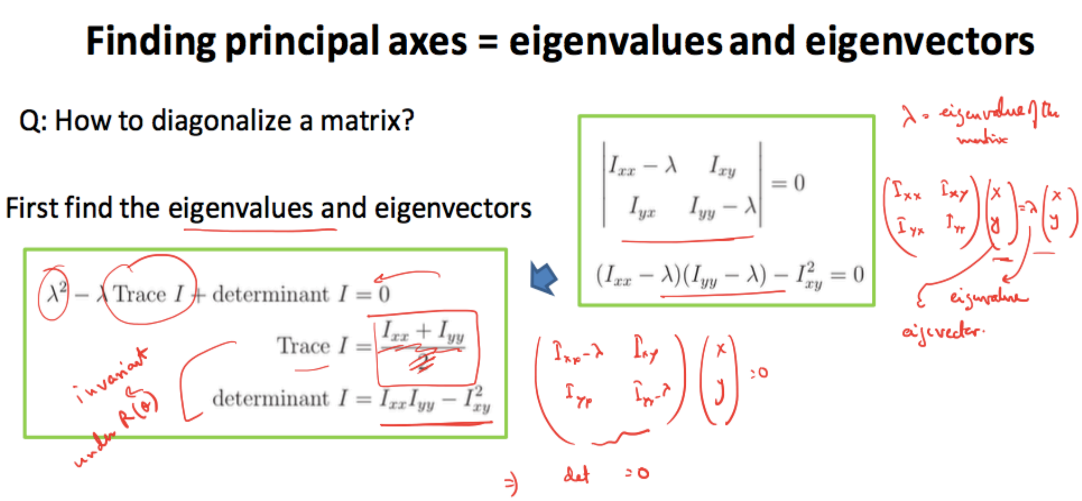
So, now this is a standard problem in linear algebra. How to diagonalize a matrix? there are two steps the first step is to find the Eigen values. So, you take this matrix and suppose lambda is an Eigen value and lambda is a matrix of the matrix. So, it satisfy this particular equation that Ixx, Ixy, Iyx, Iyy times some particular vector, which we denote as x times y is lambda times the same vector. So, this is the Eigen value, and this vector is called the Eigen vector. So, in order to get the Eigen value and Eigen vector, if we take this quantity on the right hand side.

We get Ixx minus lambda Ixy, Iyx, Ixy times lambda times x times y. So, this Iyy minus lambda times xy equal to 0 and this will be true if this determinant of this particular matrix is 0. And hence, this is condition I have written 10 here and then if you take the product and apply the determinant formula, you get this equation. Now, this equation I write in terms of word to emphasize that if you expand this, you get a quadratic equation where the coefficient of lambda is 1. Sorry lambda square is 1, and coefficient of lambda is this particular sorry so there it is a mistake.

There is no y 2, so the trace, which is an invariant under the transformation. It is a property of the matrix that if we take this transformation by rotation by angle theta trace remains the same and the other quantity will be this particular combination. Which is called the determinant of this matrix, which is another invariant. So, this trace and they are invariant under the rotation. So, then we can solve this quadratic equation and write down this lambda Eigen values so we get this quadratic equation.



So, we have two possible Eigen values and now you can see that these two Eigen values are precisely the moments of inertia of the object in a symmetric axis of symmetry, about the axis of symmetry. So, to summarize what we need to do is that so this is the first step and the second step is, for a given Eigen value, then we need to find the Eigen vector. So the Eigen vector is this particular vector. So once we find this Eigen vector, this will constitute the this will give us the direction of the so this direction now if you take an Eigen vector of magnitude 1 this will give us the direction of the principal axis of rotation. So, this is the procedure now the power of this particular method of thinking of the moment of inertia is a matrix, is that we can now easily generalize it to three dimension the only difference will be that in three dimensions, the moment of inertia is a 3 by 3 matrix instead of a 2 by 2 matrix is the only difference the rest of the procedure is the same: you take the matrix, so you choose any arbitrary.



Finding principal axes = eigenvalues and eigenvectors

Q: How to diagonalize a matrix?

First find the eigenvalues and eigenvectors

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} \\ I_{yx} & I_{yy}-\lambda \end{vmatrix} = 0$$

$$(I_{xx}-\lambda)(I_{yy}-\lambda) - I_{xy}^2 = 0$$

$$\lambda = \frac{\text{Trace } I \pm \sqrt{(\text{Trace } I)^2 - 4\,\text{determinant } I}}{2}$$

$$= \frac{I_{xx}+I_{yy}}{2} \pm \frac{1}{2}\sqrt{(I_{yy}-I_{xx})^2 + 4I_{xy}^2)}$$

→ $I_{x''}$ → $I_{y''}$

So to summarize, what we learned is that given an object 2D, 3D, 1D does not matter any dimension. Given an object that always exist a axis of symmetry, these are called the principal axis of rotation. I mean, if the object is irregular shaped, then it is not obvious by looking at what the principal axis of rotation is. So, the procedure that we learned is that you first compute the moment of inertia components along any particular arbitrary choice of axis and then You find the Eigen value and Eigen vectors of that moment of inertia. So, the Eigen vectors will represent the direction of the principal axis of rotation and This procedure is very powerful, very general, and works in all dimensions. Thank you.