Newtonian Mechanics With Examples

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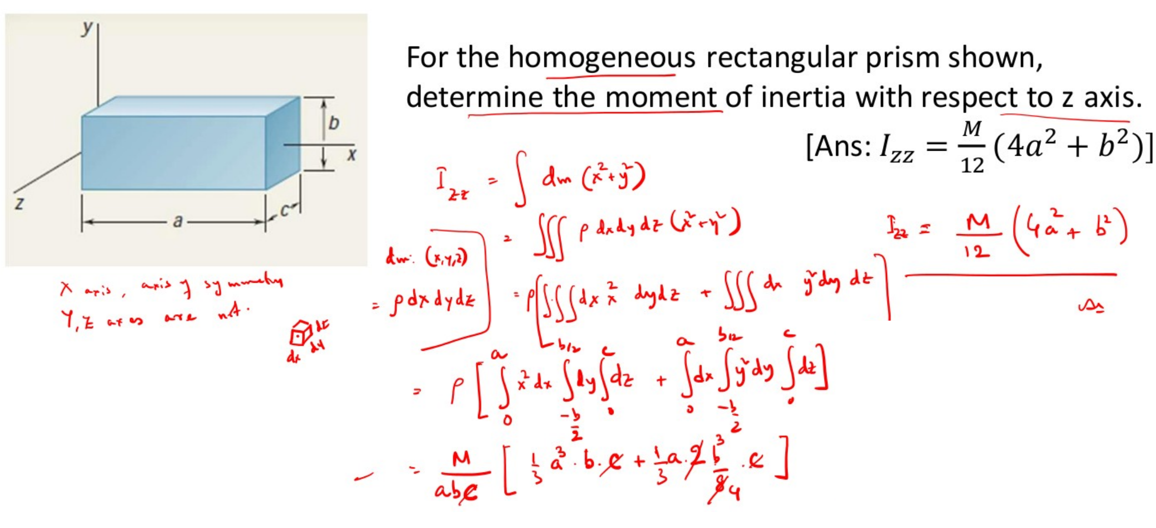
Today, we are going to introduce a new concept called product of inertia. But before that, let me quickly go through, take an example of computing moment of inertia in 3D for a 3D object. So, here is the example. So, we have a rectangular prism as shown in this figure and the question is determine the moment of inertia with respect to the z-axis. So, again note that the look at the reference frame. So, when you calculate moment of inertia, first thing is to consider what is the origin and second thing is to consider that what is the reference frame, that is, how the reference x-coordinate axis are defined.

So, the origin, so it is something, so x-axis is passing through the middle of the prism. So, x-axis is indeed a axis of symmetry for this figure, but the y-axis and z-axis are passing and not passing through the middle of the prism. So, y-axis and z-axis are not axis of symmetry, but that has nothing to worry about. So, we cannot readily use the results that we listed in our previous lecture.



But now, we do not worry about that because now we know that what is the general formula for calculating moment of inertia about any arbitrary z-axis. So, that formula is given by just we need to calculate. Now, we understand what is this moment of inertia, it is a distance of the mass distribution from the z-axis. So, we need to take each mass element dm located at some point x, y, z inside the prism and then the distance from the z-axis, so distance from the z-axis is x^2+y^2. So, now this mass element dm is if the density of the material is ρ, then and the volume of this element, so this is a little cubic cell located at width dx, height dz and breadth dy.

So, the volume is ρ dxdydz gives us the mass. So, then this is a three dimensional integral which goes from ρ dxdydz (x^2+y^2). Now, we are going to assume because it is homogeneous. So, I can take the density outside. So, this is going to give as, so I can take the ρ outside the integral and there are two terms, the first term is dx x^2 dydz and the second term which is dx y^2 dydz.

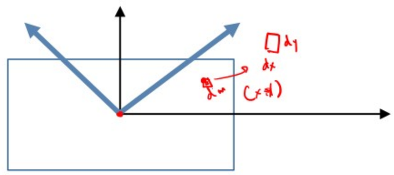
Now, this is a rectangular object, so x, y, z can vary independent of each other, so we can factorize the integral. So, we get ρ times, first term is x^2 dxdydz and the second term is integration over dx then integration y^2 dydz. Now, all that remains is to put the limits of the integral. So, from the figure, x varies from 0 to a, now y varies, be careful, so this is why you need to pay attention to the detail, the y varies from -b/2 to b/2 and z varies from 0 to c. Then we get our answer, now ρ we can write as the total mass if is M.

So the mass of the object let us assume to be M and the volume of a prism is just abc and then the first term is $(1/3)a^3bc$ and the second term is $2ab^2c/4$. So, if I simplify it, we get, so this is $b^3/8$, so this becomes 4 and here 1/3. So, then if I simplify, so the c gets cancelled from both the term. So, if I simplify we get I_{zz} is M/abc, so the first term has 1/3 and the second term has 1/4, 1/12, so you take 1/12 out, then the first term will have $4a^2$ and the second term will have only b^2 and this is the answer. So, so far the moment of inertia xx, yy and zz, this measures the spread of the mass distribution about this axis of rotation, if we choose x-axis, y-axis, z-axis as our axis of rotation.

Now, often we also want to capture another aspect of the mass distribution, which is that whether this mass distribution are symmetric or asymmetric about our chosen axis and here we need introduce the concept which is called product of inertia, which measures if the mass distribution is symmetric or not about our chosen coordinate axis. So, here let us take an example. So, suppose we have a 2D mass distribution, this is a 2D plate and this is some plate and we want to consider the moment of inertia of this plate and we have two, we consider two possible choice of our coordinate system, the blue and the black. Now, we define the product of inertia in the following way. So, we define the, so let us say any point, any mass element, so if this is our origin, then any mass element here of let us say with dx and height dy located at a point x,y.

Then we define this double integral xy times dm, where dm is this mass element. So this is a 2D, in 2D case this is the only example, only product of product component that is possible. The, the in 3D we can define two more analogous quantities, the one is the yz in a three dimensional case, yz and integral over yz which is called the component I_{yz}. So, I the yz component of the moment of inertia. So, note that in the previous case, so now this indices are different and that is why this is called product of inertia and similarly

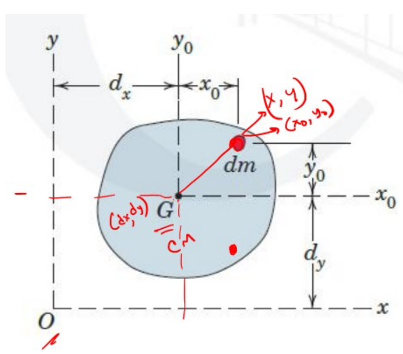
I_{xz} or I_{zx}. Now, there is a crucial difference from the previous case where we are considering the distance.



$$I_{xy} = I_{yx} = \int xy \, dm$$
$$I_{xz} = I_{zx} = \int xz \, dm$$
$$I_{yz} = I_{zy} = \int yz \, dm$$

So, this components can be, the sign of this component can be either positive or negative. To contrast, when I define the distance of a mass from the x-axis that was defined to be, so this is always 0 that is non-negative, strictly non-negative, it cannot be negative. Whereas, this combination where we have a mixed coordinates, they can. So each term has only one coordinate in this case, but here each term has a combination of x and y. So, this is a product of inertia.

Now I quickly go through the, so for the product of inertia, we can apply the parallel axis theorem which is sometimes useful. So, here is how it works. So, we consider two reference systems, one is x_0 y_0. So, G is the center of mass position and let's say if we take G as the origin and then x_0 y_0 is kind of a reference coordinate system in which the mass distribution is symmetry. And then we take an arbitrary coordinate system Ox at a arbitrary origin, location O as our origin and then the Ox and Oy at the new coordinate system in which the mass distribution is asymmetry.



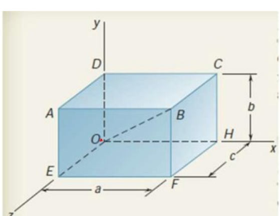
This is clear because you see that, so this x_0 axis is sort of passing through the middle of the object. So, it is clear that if there is one mass element dm here, we can also has another mass element in somewhat roughly in the opposite side. So, this is at a position which is symmetric about x-axis and similarly as a position about symmetric about y-axis. So, this is the meaning of asymmetric and symmetric mass distribution. But when we take the other axis, the full object is on one side of the axis.

So, this is clear that about Ox Oy axis the mass distribution is asymmetry. So, then we calculate the we write down the definition of the product I_{xy}. So, this is xy dm. Now, from the figure it is clear that $x=x_0+d_x$. So, d_x, so x is the location of this point with respect to the origin O and x_0 is the location of this point with respect to the center of mass.

So, and the location of the center of mass in respect to origin O is d_x and d y. So, then from the figure the x is equal to $x_0 + d_x$ and y is equal to y_0+d_y. Now, if you expand this term, you get this four term and it is easy to show that these two term will be 0. Then you end up, so this first, term is just the by definition the product of inertia with respect to the reference frame which is in the center of mass frame. So, this is by our notation we are we define it as x_0 y_0.

So, I of x_0 y_0 and the second, so you can take d_x d_y outside integral then integral over dm will give you the total mass of the object. This is just md_xd_y and then these two term you can easily show from applying the definition of the center of mass. Then these two term should vanish. So, this is analogous, so if $d_y=d_x$, then this will be the distance of the center of mass from the from this axis. So, this is the analogous of the parallel axis theorem which we so far learnt. The each axis x-axis or y-axis. Now, we we sort of applied and generalized it to the case of product of inertia.

So, now to get us familiarize with this concept, let us work through an example. So, again we take a rectangular prism,



Consider a rectangular prism of mass m and sides a, b, and c. Determine the moments and products of inertia of the prism with respect to coordinate axes shown.

Ans:

$$I_{xy} = \frac{1}{4}mab, I_{yz} = \frac{1}{4}mbc, I_{zx} = \frac{1}{4}mca$$

$$I_{xx} = \frac{1}{3}m(b^2 + c^2), I_{yy} = \frac{1}{3}m(c^2 + a^2), I_{zz} = \frac{1}{3}m(a^2 + b^2)$$

ρ = density = $\frac{m}{abc}$

$$I_{xy} = \int dm\, xy = \rho \int dx\, dy\, dz\, xy = \rho \int_0^a x\,dx \int_0^b dy\, y \int_0^c dz$$

$$= \rho \cdot \frac{1}{2}a^2 \frac{1}{2}b^2 c$$

$$= \frac{m}{abc} \frac{1}{4} a^2 b^2 c = \frac{1}{4} mab$$

$I_{yz} = \frac{1}{4} mbc$,

$I_{zx} = \frac{1}{4} mca$.

So one corner of the prism is our origin and then thus each side of the prism as x, y, z coordinate system. So, clearly in this particular choice of coordinate system the mass distribution is on the one side of the coordinate axis, so this is asymmetric. Which means that we expect that the product of inertia to be non-zero by definition. So, the question ask that determine the moments and products of inertia of the prism with respect to the coordinate system axis shown which means we need to calculate the different components I_{xy} I_{yz} I_{zx} as well as I which represents the three possible products of inertia and I_{xx} I_{yy} and I_{zz} which represents the moment of inertia above the three axis are shown.

So, let us work it out, so let us calculate the I_{xy} so our definition says, So if I take any mass element located at location x, y, z, so the x coordinate times y coordinate times the mass element that is an integrate that is that will give us the I_{xy}. So, if the density is ρ and this is constant, so I take it out times xy and this is a rectangular prism, so x, y, z are independently varying variable. So, I can factorize. So, we have xdx dyy dz and now we put limit, so x varies from 0 to a, y varies from 0 to b, so note that now y varies from 0 to b unlike the previous example and z varies from 0 to c. So, this will give us $(1/2)\rho a^2(1/2)b^2c$.

So, now I plug in replace density by the total mass,so this is one-fourth a^2 b^2 c, so upon simplification it becomes (1/4)mab and then by symmetry by switching the indices you can easily verify that the I_{yz} will be given by (1/4)mbc and I_{zx} will be (1/4)mca.

So, let us work out the I_{xx} component also, so the I_{xx} will be, so this is the distance, so this is the distance of this mass element from the x-axis, so this is y^2+z^2. So this will be ρdxdy $dz(y^2+z^2)$, so, this is really a volume integral and we can factorize x y z coordinate. So, this will be $\rho dxy^2dydz + dxdyz^2dz$ and putting the limits, so 0 to b, 0 to c, x y is from 0 to a, y y is from 0 to p, z y is from 0 to c, so this becomes, so ρ we can write as M/abc. So, after simplification it becomes (1/3) $m(b^2+c^2)$. So by switching the kind of a cyclic permutation of the indices, because this integrals are symmetric in x y z, so we can guess that I_{yy} will become (1/3) $m(c^2+a^2)$ and I_{xz} will be (1/3) $m(a^2+b^2)$

So, to summarize, today what we learnt is a new concept called products of inertia and this is a measure about whether a mass distribution, a given mass distribution is symmetric or asymmetric about the choice of our coordinate system. So if the mass distribution is symmetric then this quantity is 0, if the mass distribution is not symmetric then the product of inertia is non-zero. So in the next lecture we shall consider a new type of problem which is about principal axis and how to find them. Thank you.