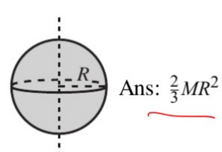
**Newtonian Mechanics With Examples**

**Shiladitya Sengupta**
**Department of Physics**
**IIT Roorkee**
**Week -07**
**Lecture-37**

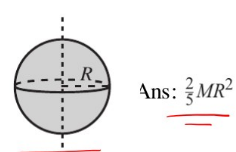
Let us continue our discussion of moment of inertia. So far we have taken several examples but all those examples were in 2D. Today let us take some examples in 3D that is we are going to talk about moment of inertia of extended objects in 3 dimension. Now the difference between 2 dimension and 3 dimension is the following. In 3 dimension there are in the most general case we have to deal with 3 independent axis of rotation. In 2 dimension if the mass distributed in only a plane such as my hand and then this is rotating in the plane then it is only possible axis is perpendicular to the plane.

However in 3 dimension there is not a single possible orientation but there are any possible orientation and in general as you know that any direction can be represented by 3 independent orientation in 3D. So we are dealing with 3 independent axis of rotation which means that there will be at least 3 components of moment of inertia. So, this is why 3 dimension is little bit more complicated than 2 dimension. So, we take some simple example of particularly symmetric mass distribution.

So let us first take a spherical shell of mass M and radius R and we take an axis of rotation as any diameter that is any axis that is passing to the center of the shell. In this case the standard so, one can calculate by applying the definition that we have discussed so far. One can easily calculate the moment of inertia. This calculation is particularly easy if we use spherical polar coordinate system which we have not discussed in this course. So I am going to skip the calculation part and give you the result and any standard textbook will contain the derivation of this result.




Ex 9) A spherical shell of mass M and radius R, any axis through center;




Ex 10) A solid sphere of mass M and radius R, any axis through center

Probably you have already seen that derivation in high school level. And then we can also take another example which is the right side is that instead of a spherical empty shell we can also take a solid sphere. This is a solid sphere of mass M and radius R. So now the shape of the object is same but it is the sphere is filled not empty. So we see that the answer the moment of inertia will be different.

In fact, the moment of inertia is now smaller compared to the spherical shell. This is reminds us of the analogous result in two dimension where we saw that a fill disc has same mass and same radius have lower moment of inertia compared to a ring. Now if you take go to any textbook you will find a lot of standard examples of different simple shape such as a rod. So now this rod is no longer one dimensional we consider that is kind of a pipe so it has some but small but finite diameter and so you have to specify now so let us say we have this G represents the centre of mass of the rod which is the middle point and then we talk we consider a reference frame x and y and z as shown in this figure which is passing through the centre of mass. Now in this particular frame the moment of inertia about the y-axis and about the z-axis are (1/12)mL$^2$.

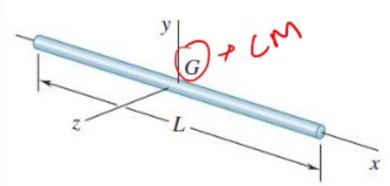
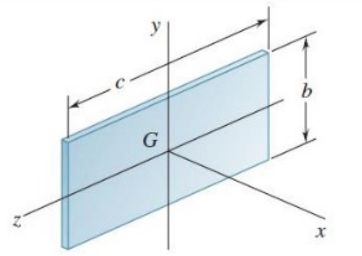
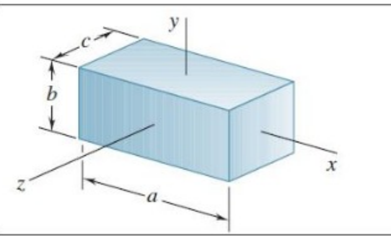
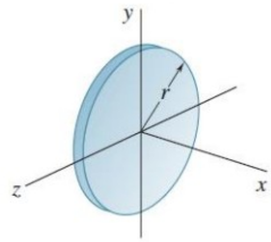
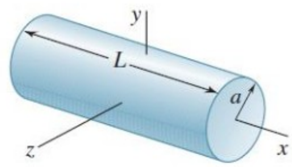
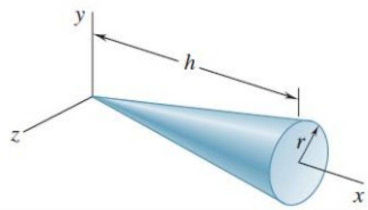
| | | |
|---|---|---|
| Slender rod |  | $I_y = I_z = \frac{1}{12} mL^2$ |
| Thin rectangular plate |  | $I_x = \frac{1}{12} m(b^2 + c^2)$<br>$I_y = \frac{1}{12} mc^2$<br>$I_z = \frac{1}{12} mb^2$ |
| Rectangular prism |  | $I_x = \frac{1}{12} m(b^2 + c^2)$<br>$I_y = \frac{1}{12} m(c^2 + a^2)$<br>$I_z = \frac{1}{12} m(a^2 + b^2)$ |

Image ref: *Vector Mechanics for Engineers,* by F. Beer, E. Johnston, D. Mazurek, P. Cornwell, B Self, McGraw Hill.

Similarly we can take a thin rectangular plate and again we are considering a reference frame where the origin is the centre of mass of the frame and then the x, y, z direction as shown and then we have some standard result. We can also take a rectangular prism, a thin disc, a circular cylinder, circular cone so some of these simple shapes that I have listed here. So, in all those cases remember that the origin is at the centre. So, there is some relation between calculation of moment of inertia and calculation of centre of mass. Often we need to know the calculate the centre of mass position first to calculate the moment of inertia.

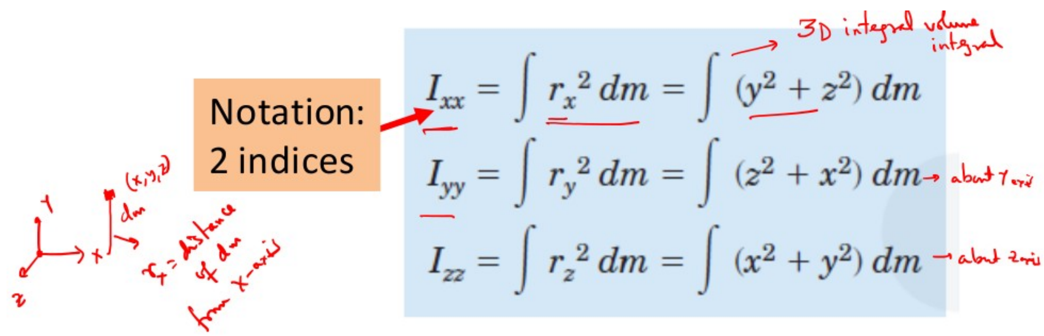
| | | |
|---|---|---|
| Thin disk |  | $I_x = \frac{1}{2} mr^2$<br>$I_y = I_z = \frac{1}{4} mr^2$ |
| Circular cylinder |  | $I_x = \frac{1}{2} ma^2$<br>$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$ |
| Circular cone |  | $I_x = \frac{3}{10} ma^2$<br>$I_y = I_z = \frac{3}{5} m\left(\frac{1}{4}a^2 + h^2\right)$ |

Now there is one thing I want to say now. So you see that here these values results are listed in the textbook. So, these are kind of available readymade. So why should I bother to calculate moment of inertia myself. So the point is that here these results are given to you readymade but they are of limited use.

Why? Because all these results are valid only for some special shape of the object first. Second for a special choice of the axis of rotation. In each of the case if you think about, we are choosing the axis of symmetry of the object as our axis of rotation. Now in practical situations in real life situation I will show you some examples. These two things are often not possible that if you are dealing with some objects which are not some simple shape as this part more complicated shape or even irregular shape then it is not easy always to immediately guess what is where is the centre of mass by symmetry, the symmetry argument fails. Similarly if I just by looking at the object it may not be possible to guess what is the symmetric axis of symmetry.

So if the symmetry argument fails then we have no way at this moment to calculate the moment of inertia. But suppose you want to calculate the moment of inertia for further analysis of the project or mechanical problem that you are having. For example you may be doing a project on robotics and there you are not dealing with some simple objects such as this. So, what we need is a method which of calculating moment of inertia which you can use in arbitrary situation. So this is something that is a powerful way of it is kind of a mindset. So, it's a powerful way of thinking so whenever you are dealing with a problem, you should not be satisfied just by knowing the solution but you must be interested to know the method by which you can yourself find the solution.

So, here is the definition of how can we generalize the moment of inertia about arbitrary axis or a three dimensional object. So I am taking three dimension because that is the most general case, but these things are also applicable in two dimension, one dimension etc. So what we have here is that I have written this as I said that in three dimension there are in general three possible axis of rotation but here I am not making any guess or any constraint that they should be axis of symmetry. They can not be an axis of symmetry arbitrary choice of axis x-axis, y-axis, z-axis. So you take any point in 3D as your origin not the center of mass, but any point then you draw some x-axis, some y-axis, and some z-axis. Then the moment of inertia the three  components are defined as the following.




$$I_{xx} = \int r_x^2\, dm = \int (y^2 + z^2)\, dm$$

$$I_{yy} = \int r_y^2\, dm = \int (z^2 + x^2)\, dm$$

$$I_{zz} = \int r_z^2\, dm = \int (x^2 + y^2)\, dm$$

So this gives you a physical meaning  now we understand what is this moment of inertia about x-axis, y-axis, z-axis. So, we want to know the spread of a mass. So we have a mass distribution some arbitrary object and we draw an our reference frame the x-axis in our reference frame and we want to know that what is the spread of that mass distribution from with respect to the x-axis, which means we are looking at not only the mass but also the distance from the x-axis. So, the combination of these two will give us the moment of inertia. So, the moment of inertia measures the spread of the mass distribution about your axis of rotation. Now note one important point I have used two indices the reason will be clear in coming few lectures. So I just wanted to highlight that often in some textbooks you will find that the moment of inertia is about x-axis, y-axis, z-axis are represented by a once single note in this like $I_x$, $I_y$, $I_z$. I prefer to use two indices to sort of so that when we discuss the our next concept the product of inertia which requires two indices hence we put two indices here.

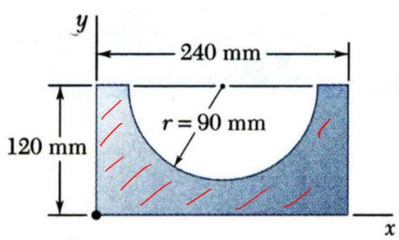
The other point is that so often in some another variation in the textbook you often find is that suppose the object has a finite density, constant density, uniform density, so then $\rho$ is the density that is if the mass of the object is M and its volume is V, then

$$I = \rho \int r^2 \, dV$$

If density $\rho$ is constant, MI is a pure geometric property

M/V=ρ and this is constant. Then this dm, remember dm=ρ dV and then you can take this ρ outside the integral and hence it will become a pure geometrical. So, the whatever integral presents is a pure geometrical property of the object and is just a volume integral. So now we see that this is even though I have shown only one integration sign but it is actually a volume integral so we are doing three integrations. So let us now take an example. In this example even though this is a example with a planar object a lot of aspects about moment of inertia calculation and various tricks will be evident, I am going to use in this example. So you pay attention to through this example. So, here is kind of a practically important example because here we want to calculate the moment of inertia not of a simple shape but a combination of simple shape.

So, the composite object. So remember it is analogous to what we did earlier to calculate center of mass. So, it is something similar here also we are going to calculate instead of center of mass location we are going to calculate the moment of inertia of the shaded the blue shaded region.

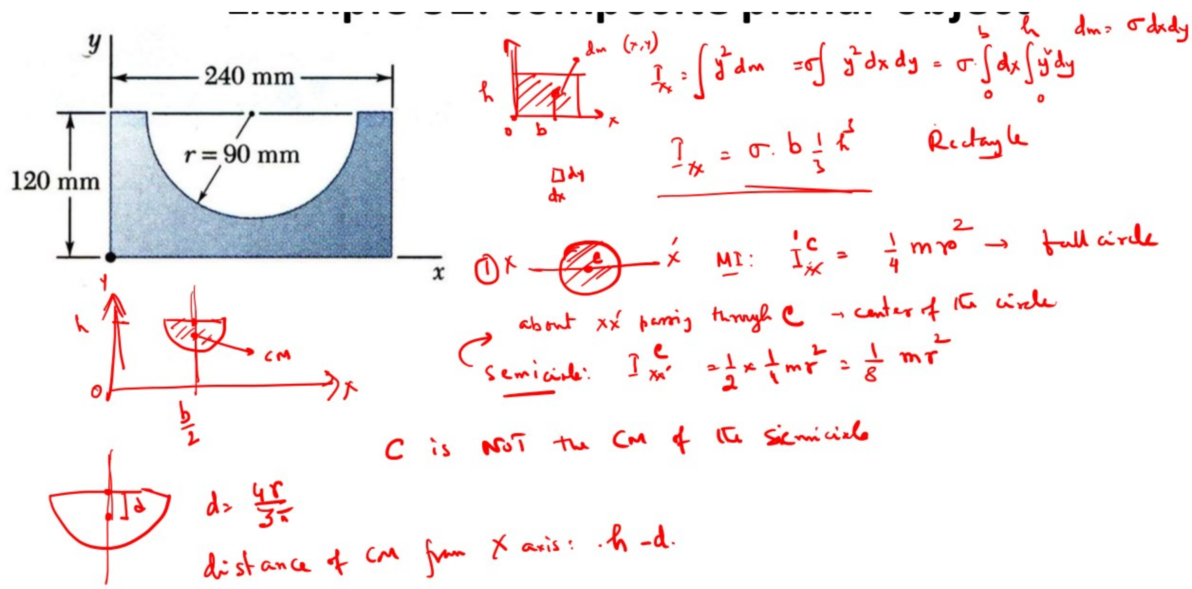



Determine the moment of inertia of the shaded area with respect to the x axis.

[Ans: $I_{xx} = 45.9 \times 10^6$ mm$^4$ ]

So, the blue shaded region you cannot describe as some simple shape but you can think of it as the rectangular plate which is a simple shape for which it is easy to calculate moment of inertia and the hemisphere. So then you can calculate the moment of inertia of this plate and then we calculate the moment of inertia of this hemisphere or semicircle and moment of inertia of this rectangular plate then we get the moment of inertia of if you subtract then we get the moment of inertia that we desire.

So, here let us compare with moment of inertia and mass so this is a crucial difference. Difference is the following. In the case of a mass, we can always take a mass and if you scoop out this portion then the mass of this rectangle minus the mass of the semicircle is going to give us the mass of the composite object as shown in the figure. However moment of inertia not only depends on the mass of an element but also the distance from an axis which means that this subtraction you can only do if you are calculating all moment of inertia in the same reference frame about same origin and axis of rotation. So, this is something you should keep in mind only then we can do this simple subtraction. So, let us proceed so first let us do the calculate the moment of inertia of the rectangular plate so this is our origin and let us say this is b and this is h so the dimensions are given so, what is the moment of inertia of this particular object. So in this case we are going to do the following so take a mass dm at a point x,y.



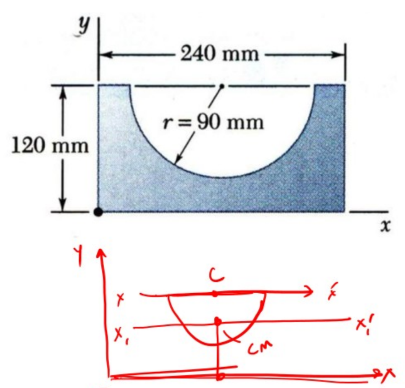
So, let us say the area density this mass per unit area is σ then and the area element so if it is a rectangular element of with dx and height dy then the area is σ dx dy. I assume σ is constant.

So, I can sort of pretend that these are basically product of two one dimensional integral and now I need to put the limit and the limit are from for x goes from 0 to b and y goes from 0 to h so we have for the rectangle. Now let us come to the calculating the moment of inertia of the semicircle so let us try it so the semicircle is something like this. So, from this point is at the point b/2 and this is h. Now for the semicircle we in our list we did not probably calculate the moment of inertia of the semicircle so note that this system of axis that is given in the problem is not an axis of symmetry for the semicircle because it is not even passing through the center of mass and of course, it is a mass distribution of the semicircle is entirely on one side of the axis so it is not symmetric about x-axis given in the figure.

So, how to calculate the moment of inertia of the semicircle, so here we can use some our tricks and our parallel axis theorem so the first thing is that instead of semicircle if we have a full circle a solid disk filled circle in that case we know and we consider an axis which is parallel to the x-axis, let us call that xx'. Now, about this axis and this field circle so this is the field circle this is the field semicircle this we know or in our result it is given so the moment of inertia so let us call that this point as C. So $I_{xx}$ about C, this we have seen before that this is $(1/4)mr^2$.

Now, if we about the same axis so this is about xx' passing through C which is the centre of the circle. So if we have a full circle centre of the circle is the centre of mass by symmetry. So then if I take now cut the upper half then for the rest of the half about the same axis we can simply say that this is just for the semicircle. So, this is for the full circle let us call that I' and this is $I_{xx'}^{C}$. Now the semicircle about the same centre and same origin and same axis will be half of $(1/4)mr^2$ by symmetry is just half of the full so this is $(1/8)mr^2$. But now C is not the centre of mass of the semicircle so it is clear that the centre of mass is no longer at this point but it will be somewhat down somewhat to the middle because this is a point about which the semicircular mass distribution is more or less kind of balanced about it it is kind of a centre of the semicircular mass distribution.

So it is somewhat lower. So let us call this point as CM of the semicircle. Now, this we have calculated the centre of mass of a quarter of a circle in one of the earlier lectures so I am not going to show you can generalize that result. I'm going to use that result but it is easy to show that the centre of mass will be located will be of course it is by symmetry it is clear that this will be along this line that is passing to the centre of the semicircle and parallel to the y-axis and it will be located at a depth d where $d=4r/3\pi$. So, this is some analogous to the result we derived calculated in some other earlier problem. So, then the centre of mass its coordinate that is its distance from the x-axis is h-d. So now if we can apply parallel axis theorem and if it says that if the centre of mass the if we take an axis so let's draw the picture again in a zoomed in way.




Apply ∥ axis theorem:

$I_{xx}^{C} = I_{xx}^{CM} + md^2 \Rightarrow I_{xx}^{C} = I_{xx}^{C} - md^2 = \frac{1}{8}mr^2 - md^2$

$I_{xx} = I_{xx}^{CM} + m(h-d)^2 = I_{xx}^{C} - md^2 + m(h-d)^2$

$I_{xx} = \frac{1}{8}mr^2 - md^2 + m(h-d)^2$ (semi-circle)

$m = \sigma \cdot \frac{\pi r^2}{2}$

$\sigma = 1\ gm/mm^2$

$r = 90\ mm$, $h = 120\ mm$

$d = \frac{4r}{3\pi} = 38.2\ mm$

$h - d = 120 - 38.2 = 81.8\ mm.$

So we have taken this axis and calculated the moment of inertia. Let us take another axis and remember that this is our original x-axis about which we are supposed to calculate the moment of inertia. Now we take another axis which is parallel to x-axis and passing to the centre of mass. So parallel axis theorem says that that means the moment of inertia about the axis $x_1x_1'$ which is passing to the centre of mass is given by the $I_{xx}^{c-}$ $md^2$ which is $(1/8)mr^2$ $-md^2$. But we are not finished because we have to our target axis is the x-axis.

So again we can apply parallel axis theorem and calculate the moment of inertia about our target axis now which is and this distance now we need to look at this distance and this distance is h-d. So if I plug in everything we are going to have our required moment of inertia of the semicircle. Well we derived some expression. So, now we have to put the numerical values. So this I leave to you as an exercise to complete this calculation and plug in the numerical values.

So I will provide some hint. So according to the problem some of the stuff are given so r is given as 90 millimetre the h is 120 millimetre. Now, d we have just mentioned this and you can show that this turns out to be about 38.2 millimetres. Then our h-d will be 120-38.2, which will be out 81.8 millimetre.

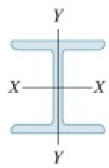
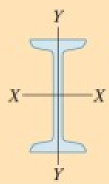
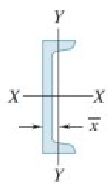
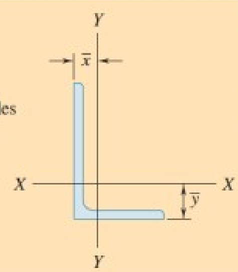
Now we need the m, m is the mass of the hemisphere. So mass of the hemisphere is if the density σ times $\pi r^2/2$. So let us assume that σ is 1 $g/mm^2$. So, take this value of σ which is the density and then you calculate the mass. Now you plug in this value to get the $I_{xx}$ of the semicircle but we are not finished yet because we want to know the $I_{xx}$ of the composite object.

That is the moment of inertia about x-axis from the composite object which is the moment of inertia of the rectangle minus the moment of inertia of the semicircle.

So we have derived the value of moment of inertia of the rectangle and the moment of inertia of the semicircle here. So take the difference and show that okay so there is a unit it should be show that you check the numerics then you indicate this whether you get this particular value.

Now, I show you some example of a typical kind of engineering machine where you apply this concept of moment of inertia is something where you have different kinds of shapes like W or S or C or angles which are used as parts of different machine and you need to know if you have different machines which are rotating with respect to each other then you need to know the moment of inertia and if so this will be relevant for those of you who will be taking mechanical or civil engineering from those disciplines.

So, you may encounter this kind of list which of different sections so these are called sections and they have different geometrical properties so it may be listed as shown here and this is the column which calculates the moment of inertia. So there are several other

| | Designation | Area mm² | Depth mm | Width mm | Axis X–X $\bar{I}_x$ $10^6$ mm⁴ | $\bar{k}_x$ mm | $\bar{y}$ mm | Axis Y–Y $\bar{I}_y$ $10^6$ mm⁴ | $\bar{k}_y$ mm | $\bar{x}$ mm |
|---|---|---|---|---|---|---|---|---|---|---|
| W Shapes (Wide-Flange Shapes)  | W460 × 113† | 14 400 | 462 | 279 | 554 | 196 | | 63.3 | 66.3 | |
| | W410 × 85 | 10 800 | 417 | 181 | 316 | 171 | | 17.9 | 40.6 | |
| | W360 × 57.8 | 7230 | 358 | 172 | 160 | 149 | | 11.1 | 39.4 | |
| | W200 × 46.1 | 5880 | 203 | 203 | 45.8 | 88.1 | | 15.4 | 51.3 | |
| S Shapes (American Standard Shapes)  | S460 × 81.4† | 10 300 | 457 | 152 | 333 | 180 | | 8.62 | 29.0 | |
| | S310 × 47.3 | 6010 | 305 | 127 | 90.3 | 123 | | 3.88 | 25.4 | |
| | S250 × 37.8 | 4810 | 254 | 118 | 51.2 | 103 | | 2.80 | 24.1 | |
| | S150 × 18.6 | 2360 | 152 | 84.6 | 9.16 | 62.2 | | 0.749 | 17.8 | |
| C Shapes (American Standard Channels)  | C310 × 30.8† | 3920 | 305 | 74.7 | 53.7 | 117 | | 1.61 | 20.2 | 17.7 |
| | C250 × 22.8 | 2890 | 254 | 66.0 | 28.0 | 98.3 | | 0.945 | 18.1 | 16.1 |
| | C200 × 17.1 | 2170 | 203 | 57.4 | 13.5 | 79.0 | | 0.545 | 15.8 | 14.5 |
| | C150 × 12.2 | 1540 | 152 | 48.8 | 5.45 | 59.4 | | 0.286 | 13.6 | 13.0 |
| Angles  | L152 × 152 × 25.4‡ | 7100 | | | 14.7 | 45.5 | 47.2 | 14.7 | 45.5 | 47.2 |
| | L102 × 102 × 12.7 | 2420 | | | 2.30 | 30.7 | 30.0 | 2.30 | 30.7 | 30.0 |
| | L76 × 76 × 6.4 | 929 | | | 0.512 | 23.5 | 21.2 | 0.512 | 23.5 | 21.2 |
| | L152 × 102 × 12.7 | 3060 | | | 7.20 | 48.5 | 50.3 | 2.59 | 29.0 | 24.9 |
| | L127 × 76 × 12.7 | 2420 | | | 3.93 | 40.1 | 44.2 | 1.06 | 20.9 | 18.9 |
| | L76 × 51 × 6.4 | 768 | | | 0.454 | 24.2 | 24.9 | 0.162 | 14.5 | 12.4 |

Ref: *Vector Mechanics for Engineers,* by F. Beer, E. Johnston, D. Mazurek, P. Cornwell, B Self, McGrawHill.

properties which we are not going to discuss now. So note that this is given in the unit of millimetre. So it is independent of the density, so we are assuming here that the density is independent and basically density is 1 the density is 1 unit. So, this is where in a real life situation and then the point is note that these shapes these are neither our simple shapes so these are more like an composite object. So, this is the practical situation which is kind of real context where you need to calculate at least those of you who are engineering students will be need will require to calculate moment of inertia.

So, to summarize what we discussed today is basically we took the most general definition of moment of inertia. So now we are kind of learned moment of inertia not only in 2D but also we generalized to 3D and we take an example of where we applied the various standard result as well as parallel axis theorem etc to calculate the moment of inertia of a composite object and we showed a real life situation of composite objects where these kind of calculations are necessary. In the next lecture we are going to introduce a new concept which is called product of inertia. Thank you.