# Newtonian Mechanics With Examples

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**Week -07**
**Lecture-36**

Hello everyone. So, this is week 7 of this course of Newtonian mechanics with examples and in this week we are discussing the some concepts. So, we have discussed the centre of mass and now today we are going to discuss another quantity called moment of inertia. So this quantity is also required to describe the rotation of an extended object. So, here is the plan for discussing the moment of inertia. So, first I am going to introduce you the definition of moment of inertia and then I will show you how to calculate moment of inertia for different objects.

Once we know what is moment of inertia, how to compute moment of inertia, then in the next week we are going to discuss the physical meaning and how we apply moment of inertia to analyze problems with rotational motion. So, this week we are going to focus on the following two types of problem. So, first we are going to assume that the mass distribution is given. So, an extended object is given.

So, this is always given. So, the mass distribution is known. And we have an axis of rotation. So, let me explain this part. So, when we talk about translational motion, we know that we need a coordinate system, a origin, a reference point from which to measure the position of the object.

So, if it is a point, this is the position of the point, even if the object is extended, you can think of it as the position of the centre of mass. Now when we talk about rotational motion, so we also need in a similar way a pivot point, a point which is fixed and about which we are describing the rotation. Now in addition to a single pivot point in the rotation, you also have an entire line which remains fixed and this line is called the axis of rotation. Now in two dimension, if the motion is in 2D, then this line is the line which is perpendicular through to the plane and passing through the pivot point. So, when the body is rotating, then this axis of rotation, it is rotating about this axis of rotation.

So, this axis of, we are going to assume in our first type of problem, the axis of rotation is also specified. Now, given those two information, we want to compute the moment of inertia. So, first we are going to recall something so that you may be familiar from your high school physics course. So, you are going to compute what is your, the simple definition of moment of inertia that you have probably familiar with. But then we are also going to introduce probably a new term to you which is called products of inertia. So, these are the different components of moment of inertia.

So, moment of inertia is going to be, is an example of a quantity that is neither a scalar nor a vector. So, the nature of the quantity will be more clear in the next week when we discuss the laws of the rotational motion. So, this is our first type of problem. The second type of problem is the following that given a mass distribution, so, we are going to assume that the mass distribution and the object is known, so it is not given. Then there is a certain special set of axis of rotation called the principal axis of rotation.

So, we are going to discuss what is it and how to find it. So, these are the two types of problem that is going to be focused for next few lectures on moment of inertia. So, let us start with a definition. So, just recall that the, if you have a, let us say if you consider a 2D objects, so the moment of inertia about z-axis, so let us consider the mass distribution is confined in xy plane. Then our axis of rotation is the z-axis and pivot point is the origin and the z-axis is the axis of rotation.

Then the moment of inertia, so the take home message is that when you describe moment of inertia, you must in your mind ask two questions. What is the pivot point? What is the fixed point or the origin about which the body is rotating? Second is what is the axis of rotation, the fixed line about which the body is rotating? You must specify these two points. So, here moment of inertia about the z-axis, it is given by this formula that for a collection of point mass, so this is the $r_i$ which is the distance of a point mass from point mass $m_i$, then its location is $r_i$, position vector is $r_i$, then the distance is given by the magnitude of the position vector which is $r_i$. So the simple definition is the moment of inertia is

Moment of Inertia (MI) around z axis

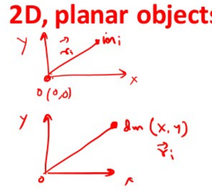
**2D, planar objects**

$$I_z \equiv \sum m_i r_i^2.$$

Collection of point masses

$$I_z \equiv \int r^2\, dm = \int (x^2 + y^2)\, dm$$

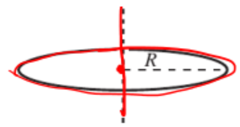
Continuous mass distribution



If you have more than one point mass, then you add up all the moment of inertia contribution from all the masses which is represented by this sum here and that there you get the moment of inertia of the total collection. If your mass is in continuous mass distribution, so for example let us say if we think of this as a flat plane object, think of the top surface only, that is a continuous mass distribution.

In that case, this sum will be replaced by an integral. Instead of a point mass, we will take some a mass element dm which is located as some x,y, that is the position vector of this mass element. But again this mass distribution is confined in the xy plane. Then you have, so this $r_i^2$ will become $r^2$ and this mi become dm and then you have to do a integral over the entire mass distribution. The sum will become an integral.

So, here is an example. Suppose you have a ring, the mass is distribution along the perimeter of a ring and the ring has a mass m and radius R. So this is the object mass distribution. Then the pivot point is the center of the circle and the axis of rotation is perpendicular to the plane. So this is the axis.




$$I = \int r^2 \, dm = MR^2$$

Ex: A ring of mass M and radius R axis through center, perpendicular to plane

And then what is the mass distribution? So it is very easy to see, if you calculate this integral which will give us the moment of inertia will be given by the total mass times the $R^2$.

So, the physical meaning of this is the following that when we have a something is rotating. So, if you look at some small region of the mass distribution, So, in the rotation, it is not only the mass, so in the translational motion, what determines, what is important so far what you have considered is the mass of a point or a mass of an object. But in the case of rotation, in addition to the mass, the another quantity which sort of decides is rotational behavior and that is the distance of the mass element, distance r from the axis. So this distance is important and that is why moment of inertia is not simply the mass but is a combination of the how far the object is, the mass is distributed in space.

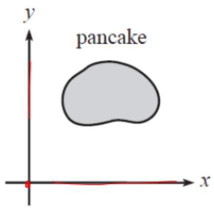
So, we have this pivot point and then what is important is that the extent of the distribution from the pivot point. This is the sort of physical meaning which distinguishes the mass and the moment of inertia. And usually how do you choose axis and this is what I am probably not discussed in high school physics that usually we choose the axis by symmetry of the object. So, for example, if we have a ring then the natural choice is an axis which is passing through the center of the ring. Just because it is clear that about the center the mass distribution will be symmetric.

However, in this course we are going to generalize this definition and we will show that you can choose any arbitrary axis and we are going to learn how to compute moment of inertia about any arbitrary choice of axis.

Next we recall the two properties, two theorems which are very useful to calculate moment of inertia. So, the first theorem is the perpendicular axis theorem and this is valid only for 2D objects which is like a pancake. So something like this a mass distribution is confined in a plane.

# Perpendicular axis theorem

This theorem is valid *only* for pancake-like objects, *i.e.* in *2D*.




Consider a pancake object in the *x*-*y* plane (see Fig.). Then the *perpendicular-axis theorem* says that

$$I_z = I_x + I_y,$$

where $I_x$ and $I_y$ are defined analogously to the $I_z$.

Image ref: *Introduction to Clasical Mechanics with Problems and Solutions,* by David Morin, Cambridge Univ. Press

Now if you consider through the same pivot point and you have the $I_z$ is the moment of inertia about the z-axis, $I_x$ is the moment of inertia about the x-axis of this object and $I_y$ is the moment of inertia of the y-axis then the theorem says that $I_z$ is going to be a combination of $I_x+I_y$. So, as long as the same pivot point this is crucial to remember. So, we are not going to discuss the proof of the theorem. We will take some examples to check its validity. But the important thing to remember that this holds only for 2-dimensional or planar object mass distribution.

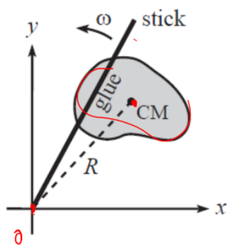
The next theorem is a more general application applicability validity. So it holds for arbitrarily shaped objects in any dimension. So, this is called a parallel axis theorem. So, suppose there is an extended object which is rotating around its center of mass.

So like this it could be a planar object, it could be a 3-dimensional object and this is the center of mass, this is the another point or a fixed point pivot point. Now what you do is that you put a stick through this object and then glue the one end of the stick at this point. So, then this point is going to be a pivot point and the object can rotate about its center of mass as well as about this origin, about this pivot point because of the stick. And consider the special case that the speed of rotation about the center of mass is same as the and the speed of rotation about this origin. Now you can sort of visualize in this case that in this particular case, this is a special case in which all points in the object travel in circles around the origin with the same angular speed.

So, I invite you to sort of do this experiment at home. So make some object through like a piece of paper and put some stick through it and then you put one end and rotate this paper and you can easily verify it by yourself. Now if in this particular case, we have two choices for the axis of rotation. Let us say our axis of rotation is a z-axis. Now the z-axis we locate at the pivot point at this point at the origin.

So, z-axis which is passing through the origin and we also take a parallel axis which is passing through the center of mass and we compute the moment of inertia of the object around these two axis, one through this origin at this point which is the moment of let us say the moment of inertia is $I_z$ and the other one about the axis parallel axis passing through the center of mass, let us call it $I_z$ CM. Then the parallel axis theorem says that

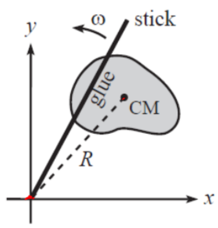
This theorem holds for *arbitrary shape objects*, *i.e.* in any *D*




Suppose an extended object rotating around its CM. And the CM rotating around the origin.

Simplified case: rotation speed $\omega$ around CM is same as rotation speed of CM around origin. E.g. glue a stick across the object and pivot one end of the stick to origin. → all points in the object travel in circles around the origin with the same angular speed $\omega$.

Image ref: *Introduction to Clasical Mechanics with Problems and Solutions,* by David Morin, Cambridge Univ. Press




Simplified case: all points in the object travel in circles around the origin with the same angular speed $\omega$.

Then the moment of inertia around the origin is

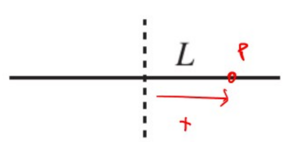
$$I_z = MR^2 + I_Z^{CM}$$

**Must pass through CM**

Mass of object

Again we are not going to prove this theorem but we are going to use this theorem in some examples. So, here is an example.

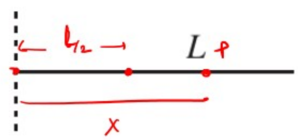
So suppose we take a stick or a line as our object, as a mass distribution and the density is uniform, so then it is easy to see that the center of mass must be at the middle point of the line, by symmetry.




$$I_{CM} = \int x^2 dm = \frac{1}{12} ML^2$$

L = length of rod

And then if you apply this definition, so if you take any point and this distance from the center of mass is x, then the moment of inertia about the center of mass and consider an axis which is perpendicular to the line, then the moment of inertia about the axis passing through the center of mass is going to be (1/12)$ML^2$ , where L is the length of rod. Now if we take the, let us say another axis which is parallel to the first axis but let us say we shift this dotted axis and put it at the end of the rod. And then if you take the same point P, now the distance about the pivot point has changed, this is now the x, the coordinate. So this coordinate and this coordinate are clearly different. So then, if you do the, calculate the center of moment of inertia about an axis passing through this pivot point at the end of the rod, then the answer is (1/3)$ML^2$ which is different from the moment of inertia.



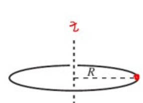

$$I_e = \int x^2 dm = \tfrac{1}{3}ML^2$$

$$I_e = I_{cm} + M\left(\frac{L}{2}\right)^2$$

But you can easily verify that the, so the distance between the pivot point at the end of the rod and the, from the center of mass is L/2. So we, you can easily check that this particular case, these calculations satisfy the parallel axis theorem.

Now we are going to take several examples, some standard examples. What do I mean by standard examples? It means that we are going to take simple geometrical object, simple shapes like a line or circle and so on. And in this case we can easily calculate, we can easily calculate this integral and we know the value of the expression of the center of moment of inertia.

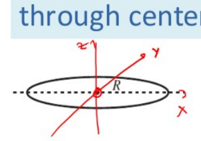
So the first example which we already discussed before is a ring of mass M and radius R and in, note that in all these cases you have to describe the object as the first step and what is the shape and its mass. Second is you have to mention the pivot point and the axis of rotation about which you are going to calculate the moment of inertia. Only then this formula will have some meaning.




$$I = \int r^2\, dm = MR^2$$

Ex: 1) A ring of mass M and radius R axis through center, perpendicular to plane;

Ex: 2) A ring of mass M, radius R, axis through center, in plane;




$$I = \int r^2\, dm = \tfrac{1}{2}MR^2$$

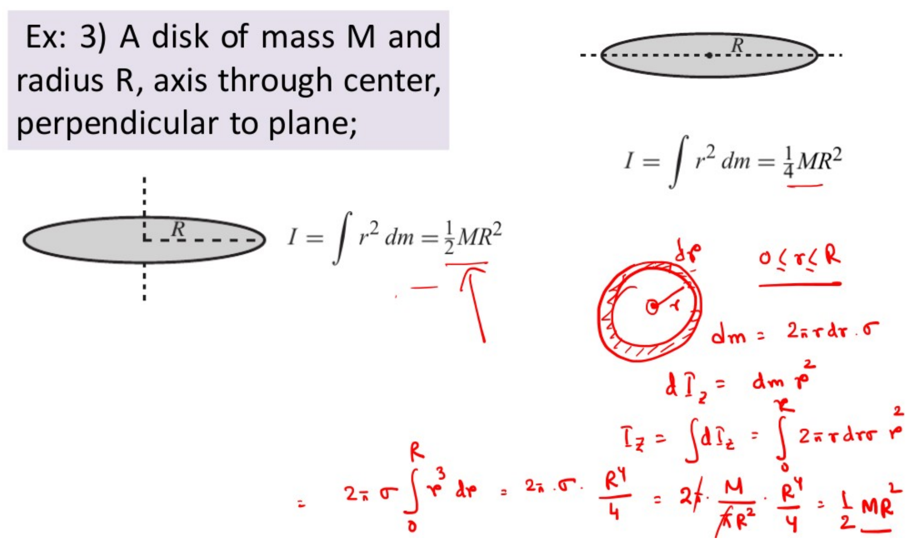
$I_x = I_y = ½\, I_z$

$I_z = I_x + I_y$

Now instead of the axis which is perpendicular, if we take in plane axis such as this, we have and then calculate the moment of inertia, we get $(1/2)MR^2$ . So this we can use to verify. So if we take this as our x-axis and another perpendicular in plane axis as our y-axis and then, sorry, say let us say this is a perpendicular axis, this is y-axis and let's say this same as this, this our z-axis.

So we can compute by symmetry,  we can choose any diameter of the ring as our axis passing to the center  and we will get by symmetry, we will always get $(1/2)MR^2$. So, $I_x$ and $I_y$ will be same,$(1/2)MR^2$ and that is going to be half of the $I_z$. So, that verifies the perpendicular axis theorem.

Your third example is to take a disc.  So now we filled up the ring and it is a disc, so a filled circle. Again the mass is M and radius R and we are going to take z-axis which is perpendicular to the plane and in this case, we will get the answer, the moment of inertia to be $(1/2)MR^2$.




And you can also do the moment of inertia about the axis which is passing through in plane axis,  let us say x and y-axis and then you shall get $(1/4)MR^2$.  So, you can also apply perpendicular, the perpendicular axis theorem to arrive at this result. So, let me point out one thing, how to, so you can sort of derive this result by starting from this ring. So if I take this ring, let's say I take a ring which is of radius r and width dr, so this thickness is dr and this is the center of the circle. Now, if r varies from 0 to capital R, then we generate this disc of the mass distribution.

And let us say that we are going to calculate the axis is z-axis which is perpendicular to this plane. So this is our ring shaped mass distribution. So for this, we are going to get the mass of this ring. Because the thickness is small, so the mass is small, so, this is going to give the area which is $2\pi r\,dr$ times the density which is let us say σ. So, this is the mass and we have already derived this result that this is going to give you $MR^2$, so the moment of inertia about the z-axis due to this particular mass distribution is going to be dm $r^2$. So now if I now vary r and get, generate this shape, then I am going to get the full moment of inertia due to that entire disc.

So this is one way that if you know some simple object in 1D and then if you use that result to get a kind of object of revolution in a higher dimension or from one result how you can get moment of inertia, derive the moment of inertia of a slightly more complicated object or slightly more complicated mass distribution. So, here I just wanted to mention one thing that what we find is that the ring has a moment of inertia which is larger than a disc.

Now this I just wanted to mention because there is some of you may have played this game, there is a game called ring, so flying ring or a frisbee which is a disc. So the flying ring is an example of a mass distribution which is concentrated on a ring and frisbee shown here is the disc separate mass distribution. So what we just find is that if you take the same mass and same radius of a ring and this frisbee then the moment of inertia of the ring will be higher compared to that frisbee. Now we shall discuss it in the next week that for the same mass and radius and also the angular speed the kinetic energy and the angular momentum will be proportional to the moment of inertia. So that means that this disc will have less kinetic energy and less angular momentum than a flying ring.

Example of ring: Flying ring

Example of disk: Ultimate frisbee

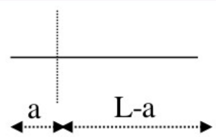
So, that is why it is much easier to control a frisbee compared to the ring. So, now let us take, so that was about some real life example from a game.

So, now let's take some more standard example of moment of inertia of extended objects. So again this we have just discussed that if you take a thin uniform rod of mass M and length L and the axis is through the center which is the center of mass by symmetry and perpendicular to the rod then the moment of inertia is $(1/12)ML^2$ and if you put it at one end then it is going to be $(1/3)ML^2$.

Let us generalize these results. So let us say that we are going to put the moment of the axis at any point a from the end, where a is a distance from one end of the rod. So when a is 0 then we get this situation where the axis is passing to a end of the rod. When a is

L/2, we get this situation where the axis is the axis of symmetry. So, the calculation is shown here and the result is, I just focus on the result, where ρ is the density that is mass per unit length and you get the moment of inertia as expected which is now a function of a. So I put it as an exercise that you plug in a is equal to L/2 and show that you get back the result and a is equal to 0 you get back the result discussed earlier.

Ex: 7) axis through x=a, perpendicular to rod




$$x' = x - a$$

$$I = \int_{-a}^{L-a} x'^2 \, dm$$

$$= \int_{-a}^{L-a} x'^2 \rho \, dx$$

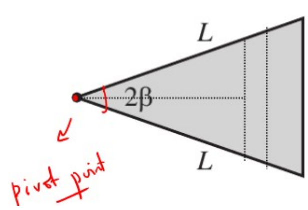
$$I(a) = \frac{1}{3}\rho \left[(L-a)^3 + a^3\right]$$

$$\rho = \text{density} = \frac{\text{Mass}}{\text{length}}$$

So, the take home message from this example is that the choice of origin and the choice of axis is absolutely important to determine the moment of inertia. So it is meaningless to talk about moment of inertia unless you choose your axis and choose your origin, the pivot point.

Now let us take another interesting example.

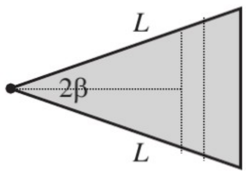
Ex 8) Isosceles triangle of mass M, vertex angle 2β, and common-side length L, Axis through tip, perpendicular to plane




$$I = \int dm \left(\frac{\ell^2}{12} + x^2\right) = \int_0^h (\rho 2x \tan\beta \, dx)\left(\frac{(2x\tan\beta)^2}{12} + x^2\right)$$

Now we are putting the axis which is through the, this particular vertex and the axis is perpendicular to the plane. So this is our pivot point. So this is the pivot point. Then how much is the moment of inertia of this particular mass distribution?

So I have, it is worked out in detail. So, I am going to skip the intermediate derivation, but come straight to the final result. So, the final result is this following. So this, obviously this moment of inertia depends on the angle β. So it is kind of intuitively obvious. So determines the spread of this mass about this pivot point. So if you change β, you expect that the moment of inertia will change.




$$\lambda = \frac{M}{A} = \frac{M}{(\frac{1}{2}h(h\,\tan\beta))}$$

$$dm = \lambda(2\,x\,\tan\beta)\,dx$$

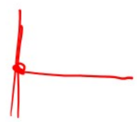
$$dI = dm(\frac{(2\,x\,\tan\beta)^2}{12} + x^2)$$

$$I = \int_0^h dI$$


$I = ML^2/2$

$$h = L\,\cos\beta$$

$$I(\beta) = \frac{M\,L^2}{2}(1 - \frac{2}{3}\sin^2\beta)$$



$$I(0) = \frac{M\,L^2}{2} \text{ not } \frac{M\,L^2}{3} \text{ which is the rod result}$$

So what I want to point out is that, naively we may think that if we put β=0, that means if we reduce this angle and make it almost approaching 0, then geometrically this object is going to be like a rod. So, if you close this angle, it is going to be a rod. But if we plug in the value β=0, we are going to get the moment of inertia of this triangle to be $ML^2/2$ . Now we have just calculated or discussed the moment of inertia of a rod. So this, if we take this particular situation where $\beta \to 0$, this is kind of a rod and we are calculating the moment of inertia, this is the pivot point and we are calculating the moment of inertia about the axis perpendicular to the rod at the pivot, passing through the end of the pivot point at the end of the rod.

But that moment of inertia was $ML^2/3$. So they are not matching. So that means, what we see is that if I take a planar triangular mass and close this angle, geometrically we arrive at the one dimensional rod, but they have a different moment of inertia. Now this is interesting and why is it so? So here is the answer that it is not straightforward to reduce a triangle to a line because a triangle has a constraint, the sum of all three angles has to be 180 degrees. Because of this constraint, if you simply reduce β=0, we will not get a rod. So, the previous case I told you that you can use one object and use it to calculate the the moment of inertia of another object, but the take home message from this example is that you must be careful.

So to summarize what we discussed today are several examples of simple shape objects and calculate their moment of inertia and we took how to calculate, essentially this is a calculation of integration and but you have to be careful about doing the integration about setting the limits. And first thing you should remember is that what is the pivot point about which you are calculating the moment of inertia. Second thing you should ask, what is the fixed line or axis of rotation about which we are calculating the moment of inertia. Because when we define the moment of inertia, the distance is measured with respect to this axis of rotation. So these two are very important, and without and if you change them, then the moment of inertia of the same object can be different.

So in the next lecture, we are going to consider three dimensional objects. Thank you.