Course Name: Newtonian Mechanics With Examples

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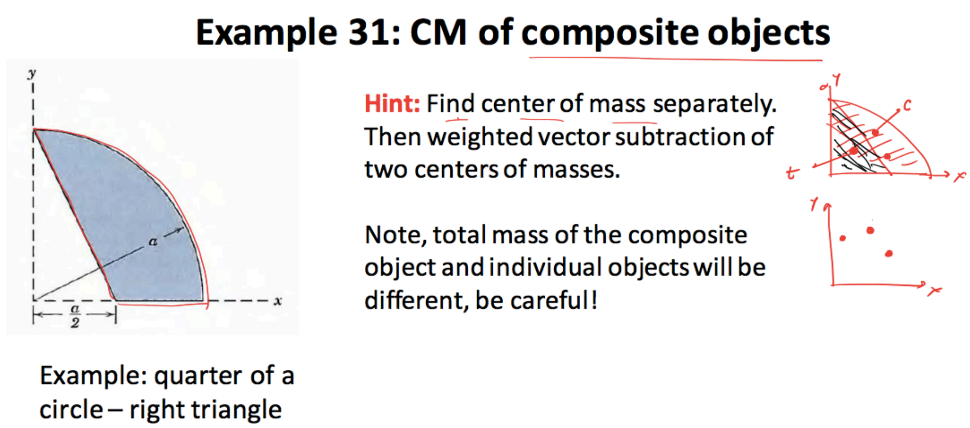
Week 06

Lecture - 35

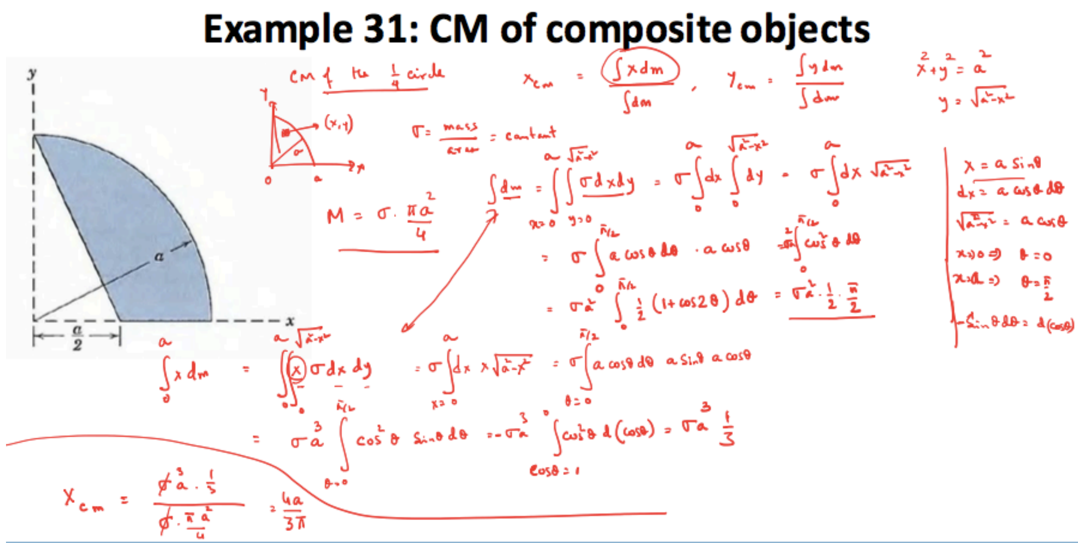
In the last lecture, we were discussing extended objects and, in particular, We are now trying to study how to analyze problems with rotational motion. And in the last lecture, we this introduced this concept of center of mass and we took several examples of an extended object with a certain mass distribution, then Where is the location of center of mass and how to find that. Today we are going to take an example, so this is an interesting example in which this is more kind of probably realistic example. So today, we are going to discuss how to find the center of mass of a composite object. So composite objects, I am going to say that this is something where the object has some shape which is not a regular shape. So in the last class, we discussed simple shapes like a straight line, a circle or a disc or a square or a triangle, and so on.

But often we in come across objects whose center of mass we want to find but We cannot describe the objects with such a simple shape. However, if we are lucky, we can still describe the shape of the object as a combination of simple shapes and this is what I mean by composite objects. So here is an example: the shaded area in the blue-shaded region represents a mass distribution. So this is where the mass is located and so this is the shape of the object and we are interested to identify the location of the center of mass that is a problem.



As we know, we can start with the basic definition of center of mass and find just by first principle calculation, which is most general and most of the time the only practical way. But here we can notice the following: the shape of the object, which we can sort of describe in the following way:. So we have a quarter of a circle, and then we have a triangle inside the circle. So now if we scoop out this triangular area, this triangular portion from the circle, then we get the given mass distribution. So in this case, we can follow the strategy that it is easy to find the center of mass of a triangle and the center of mass of a quarter of a circle.

So then our strategy will be to find the center of mass of the triangle. So let us say the triangle has center of mass somewhere here and let us say The full circle will have center of mass at the in the dislocation. So this is the circle, this is the triangle, and this particular shape, let us say, the center of mass is here. So then, if we replace this triangle by a point mass, this is x, this is y, this is x, this is y. So this is the center of mass of the triangle and this is the center of mass, which is unknown of the shaded region and then if we combine these two center of mass, we know we should.

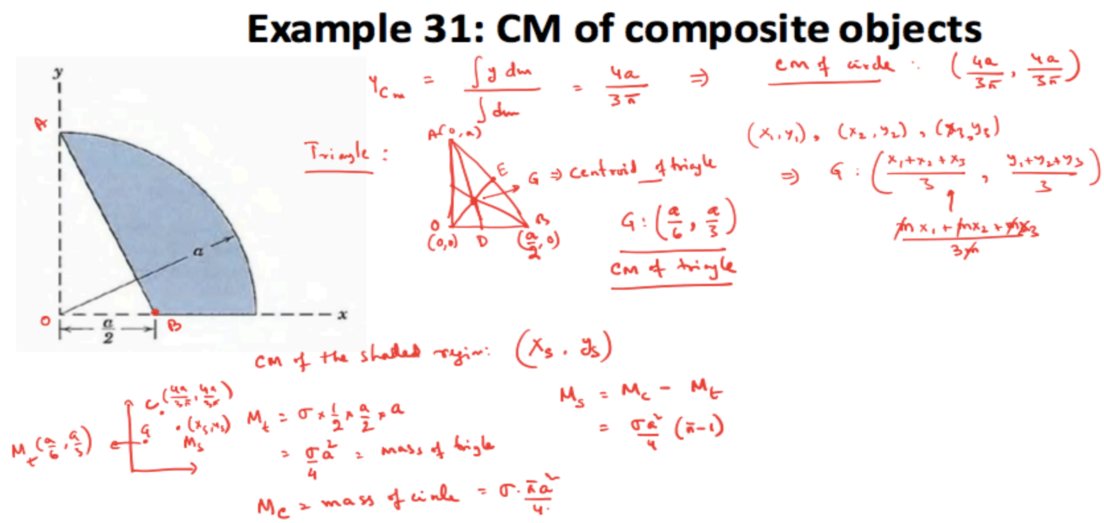


So now you think of this triangle as a single point and the shaded region as a single point located at the center of mass. Now if we combine them, we know that we can define the center of mass of this combination and we know that this must coincide with the center of mass of the full quarter of the circle because if we add the triangle and the shaded region, that makes a quarter of a circle. So this is going to be our strategy. Now I am going to work out this problem in some detail and sort of show you the algebra in detail. There are several interesting aspects about this algebra.

So let's start doing this calculation. So first, I am going to look at the center of mass of a quarter of a circle. So we have this circle—one-fourth of a circle. So according to the picture, it has a radius of a. Now if we write down the definition, if you recall the definition, consider a particular mass element shown in this square block.

Let us say that this is our origin, O is our origin and these are the coordinate axis and then assume that this mass element is located at the points x, y which are inside this circle. Then our definition, and then let us say the sigma is the density of the area density, that is, mass per unit area, and I am going to assume that this mass distribution is uniform. So this sigma is a constant. Then we can write by definition that the center of mass of if I take the location of x,. Its x coordinate is x, and let us say its mass contribution is dm.

Then, if we add over all such elements and then divide by the total mass, that will give us the x coordinate of the center of mass and, similarly, the y coordinate of the center of mass. So our first task is to calculate these integrals. Now for a circle, these integrals and the natural coordinate system will be two-dimensional polar coordinate system. But since we are, we have not introduced them in this course, so we are going to use Cartesian coordinates and do this calculation. So let us first do the calculated total mass.



So we know that the total mass will be; let us say the total mass is m of the quarter of a circle, then m must be the density times the area of the circle. So the area of the circle is pi, the radius, and this is a quarter of a circle, so the full circle is divided by 4. So we know the answer. So, now we are going to do this integral to get the answer. So this will give us some practice in the calculation.

So this integral can be written as dx dy, so this is going to be a two-dimensional integral or an area integral. So here, I am going to write two variables—an integration variable for that. Now let us think about the limits of the integral. So for the x, the limits are easy; it goes from 0 to a, as in the picture. So x equal to 0 to a.

Now, the y goes from the x axis to the perimeter of the circle. So the perimeter of the circle is given by the equation of the circle, which is x square plus y square equal to y square. So y is equal to a square minus x square. So the y goes from 0 to a square minus x square. And since the limit of y depends on x, we have to do the y integration first.

We cannot switch the x and y orders of the integral. So if I do the, so I can take the sigma outside the integral. So I can write as, and x goes from 0 to a. And if I do this integral, I end up with a one-dimensional integration, now only in x. Now to do this integral, let us recall that, Let us change the variable that lets us call x equal to a sin theta, and then dx is equal to a cos theta d theta.

So this square minus x square is equal to a cos theta. And when x equal to 0, that means that sin theta is 0, that means theta is 0 and x equal to 1, that means sin theta is x equal to a, then that means sin theta is x by a, which is 1, which means theta is equal to pi by 2. So in our new variable, this is going to be, so you have dx is a cos theta d theta times a cos theta 0 to pi by 2. So this is going to be sigma times, so I can take a square outside; we have a cos square theta d theta 0 to pi by 2. Now this cos square theta d theta 0 to pi by 2, so this is equal to sigma a square 0 to pi by 2 half 1 plus cos 2 theta d theta.

So you can check that the second term, the integration over cos 2 theta term, will give you 0 from 0, when the limit is 0 to pi by 2, and the first term will give you the contribution of pi by 2. So we get sigma a square times half times pi by 2, and this you can compare with this expression that they are matching. So we need to come to our main focus, which is to calculate this first integration. So in the first integration, new integral in the numerator which is going to be x times the mass, again x goes from 0 to a. Now, again, this is going to be a two-dimensional integration because the mass is given by this expression, so this is a two-dimensional integration.

So note that this integral is very similar to the earlier integral, except for the factor of x present here. So, I am going to write it as take sigma out, so again, we have to do the x y integration first, and the y integration we know will give us this factor of a square minus x square. So now it is a one-dimensional integral in x. So again, we play the same trick, so we use this variable change, and If we use x equal to a sin theta, then we get, so dx is going to be a cos theta d theta and x is a sin theta, and the square root of x square minus x square is cos theta. Now this goes from 0 to, so this is x equal to 0 to pi by 2, this is theta equal to 0 to pi by 2.

So now if I do the simplification, we get sigma times a cube, so note a cube has a dimension of length to the power 3 and this makes sense because we have x, dx and dy. So this will have, so this object this integral will have a, so and hence you get this dimension of a length to the power of 3, so far we are consistent. So the rest of the variables are, so you have a cos square theta sin theta d theta. Now, sin theta d theta is nothing but d of cos theta, so we can write it as, so this is still theta equal to 0 to pi by 2, but now we take cos theta as the variable, and this is going to give us something like cos square theta d cos theta, where cos theta varies from 1 to 0. Because when theta equal to 0 cos theta is 1, and when theta equal to pi by 2, cos theta is 0.

So this gives us like a 1 by 1 third. Sorry, there is a minus sign here, so there should be a minus sign here and then we can observe the minus sign by reversing the limit of the integral, so this is going to give us sigma a cube times 1 by 3. So now we have calculated the numerator and denominator, so then our x cm is going to be sigma times a cube times 1 by 3 divided by sigma times pi a square divided by 4, now, if we do the simplification, we get sigma cancels, so this is beauty that even though we introduced an unknown variable, sigma, but we are going to see that as long as our mass distribution is uniform, and the density is constant, the density can be cancelled. So this cube by a square will give us a, and the other factors will give us 4 a by 3 pi. So you can do the same way you can calculate the y coordinate of the integral and so I leave it to you as an exercise to show that the y will have the same value.

So this is also 4 a by 3 pi, so that give us the location of the cm of the circle, Which is at 4 a by 3 pi, 4 a by 3 pi. Now our next step is to find the location of the triangle. So the location of the triangle, location of the centre of mass of the triangle, now the centre of mass of the triangle is located at a point called centroid of the triangle, so this is a standard result of geometry, so we have a triangle, which is right circular triangle. Let us call this our origin, so this is 0, and at this point, let us call that O. Let us call that B and this point A, so this are O, A and B, and from the picture, You can see the coordinates of O, A, and B are the following.

Now let us show that the centre of mass of triangle will be located at the centroid, so what is the centroid? The centroid is the point that, if you take the middle point of any side of the triangle and then you connect the opposite vertex to the middle point and then do it for the another side and do it for the third side, It can be proved that all these points should pass through the same point. They intersect each other precisely at the same point, and this point is called the centroid of the triangle. Now let us give a physical argument. So this is a kind of geometrical proof that you have I am sure you are familiar with from your high school course, so let us look, at give a physical argument to show that this must be the centre of mass. So, since the triangle is uniform mass, the mass density is uniform, so if I say this, let us take this point: This line AD is going to divide the triangle into two pieces, the left piece and the right piece.

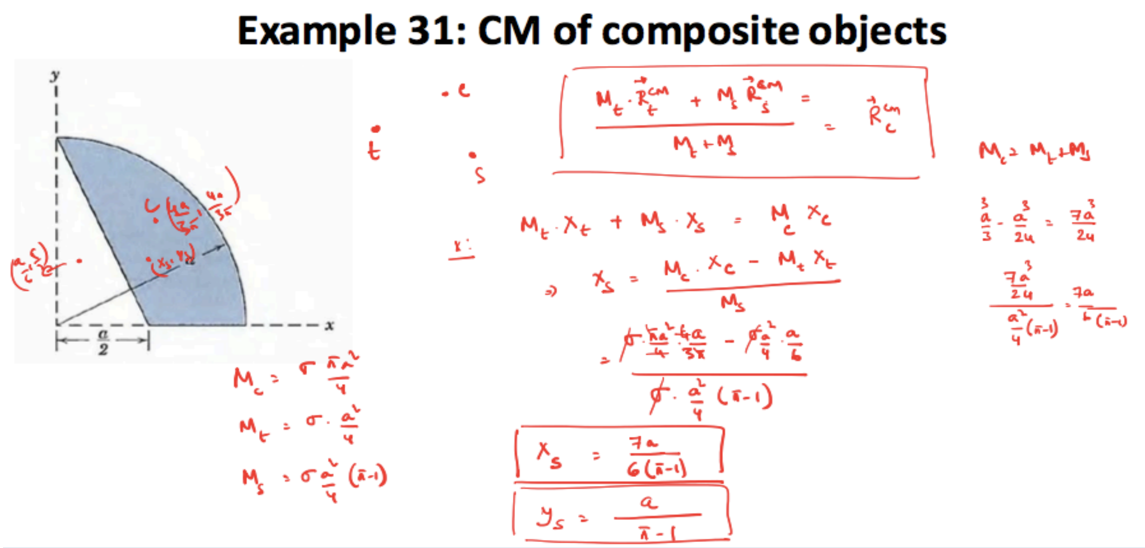
And by symmetry, if the mass density is uniform, then It is easy to see by symmetry that these two piece will have equal mass, that means the centre of mass. So, this will give you a kind of physical understanding of the center of mass. So, the centre of mass is a point which is kind of balanced, so we are going to replace the mass distribution of an extended object by a single point, so intuitively, we expect that the mass distribution of that object must be symmetric or balanced in some sense. about the centre of mass. So if we follow that line of reasoning, then we can immediately see that the centre of mass must lie on the line AD somewhere on the line along on the AD.

Now we can do the same thing with this line OE, so the line OE must also divide the triangle into two pieces. Let us call that the up piece and the down piece and again, these two piece must have same mass, so that means the center of mass must also lie on the line OE, which means that the centre of mass must be the point of intersection of the line AD and OE and this is precisely the geometrical definition of the centroid of a triangle. Now, in the interest of time, I am going to use the identity of the centroid. If the vertices of a triangle are located at these three points, then the coordinate of the centroid of the triangle is given by this beautiful relation and this you must have seen in either geometry or coordinate geometry of high school. We can sort of see it immediately by applying the centre of mass definition.

So if G is the centre of mass, now if you place three equal mass of size m, any size arbitrary mass at the three locations, then the centre of mass of that three vertices will be, by definition, mx1 plus mx2 plus mx3 divided by 3, and the m will cancel. We see that the coordinate of the centroid must be given by this beautiful formula. So you are going to apply this formula and in this case, the coordinates are given here. So, that means the location of the centre of mass of the triangle will be given by 0 plus A by 2 plus 0, which is given by divided by 3 so this is A by 6 and the y coordinate will be 0 plus 0 plus A divided by 3, which is going to be A by 3. So this is the centre of mass of the triangle.

So now we can replace this, so let us say the centre of mass of the shaded region is xs, and ys, and xs and ys are unknown. So what our strategy is going to be is that we replace the triangle by the center of mass. So this is point G, which is A by 6 times A by 3, and it has some mass m, which is equal to the mass of the triangle. So how much is the mass of the triangle with the same density? That is easy to find. This is going to give you this, which is given by the density times the area of the triangle.

Which is half into base into height, which is given by sigma by 4 A square and similarly, let us say we are going to put the shaded region so the shaded region. We do not know where this location is, but let us assume this is here so this is the position of xs and ys and this has some mass of the shaded region and then we are going to replace the triangle. So, quarter of the circle is the circle centre of mass of the circle and the circle has a mass, so this is the mass of the triangle if it were present. Which is the mass of the circle we just calculated, so this is sigma times pi A square by 4. So then we can see that the mass of the shaded region must be the mass of the circle minus mass of the triangle, so this is going to be sigma A square by 4 pi minus 1.



So now we have to calculate the centre of mass location so again, we are going to use this definition so this is the centre of mass of the triangle and this is the centre of mass of the shaded region. This is the centre of mass of the circle. So the centre of mass of the triangle. so Mt times, let us write it in a vector notation so let us call it R triangle centre of mass of R triangle plus M shaded region times the position of the center of mass of the shaded region must be equal to or divided by the total mass of the Which is nothing but that of the circle; this must be equal to the centre of mass of the circle. Now if I look at the x coordinate of this equation, we get MT times x triangle plus MS times xs is going to be MC times x c so MC is MT plus MS so then our unknown is xs.

So xs is going to be MC times xc minus MT times xt divided by MS. We have found that sigma times pi A square so let us note down the values MC is sigma times pi square by 4 MT is sigma times A square by 4; and MS is sigma A square by 4 into pi minus 1. We have our xc. So the xc was so centre of mass of the circle. So this is located at 4 A by 3 pi 4 A by 3 pi so this is the c this is A by 6 A by 3 and Let us say these are our xs, and ys.

So times A by 4 A by 3 pi minus MT is sigma A square by 4 times x t is A by 6 divided by sigma times A square by 4 into pi minus 1. So sigma gets cancelled so we have this 4 gets cancelled, and pi gets cancelled. So, A cube by 3 minus on the numerator, we have A cube by 3 minus A cube by 24. Which is 7 A cube by 24, and the numerator we have A square by 4 so 7 A cube by 24 divided by A square by 4 is going to be A divided by 7, A divided by 6. We have a factor of pi minus 1 here so pi minus 1 so the answer is 7 A by 6 pi minus 1.

So this is the location of the centre of mass x coordinate of the centre of mass of the shaded region. So now I put it as a take-home exercise and you can start from this equation. And look at the solve for the y coordinate of the shaded region and show that this is going to be A by pi minus 1. So to summarize what we have discussed in the last couple of lectures, is that how to find it if you take an extended object and if we want to analyze its rotational motion, then sometimes we need to know the centre of mass location, which is a representative point, a single point about which the mass distribution is kind of balanced or symmetric and our goal was to show how to give, so the type of problem is the following: Given the mass distribution, given the the object of the shape, and given the density of the object. Then use the definition of the centre of mass and identify its location.

So, this is mainly an exercise in calculating the integrals, so we gave several known results of simple shapes such as line rod a circle ring, etc. And for practical purposes, we also discussed something called composite objects, Which are made of combination of this simple shapes and how to find the what is the strategy to find the centre of mass of composite objects. So, in the next lecture, we are going to look at another very basic quantity required to analyze the rotational motion of an extended object. So we are going to talk about moments of inertia. Thank you.