Course Name: Newtonian Mechanics With Examples
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Lecture - 34

Today, we are going to start a new topic. So, this is where we are going to start talking about the translation and rotation of rigid bodies. So, so far in this course, the examples that we have discussed were mostly almost all the examples were: in some cases, we took point masses; in some cases, we took extended objects. But in most of the cases, we are talking about the translational motion of the object of our system. Now we are going to focus on the rotational motion. In addition, we are now going to look at extended objects, the objects which are no longer a point mass but a mass distribution which is extended and distributed in space over a certain region.

Now, rotational motion has a few counterintuitive concepts that are slightly different from translational motion. So, I am going to point higher emphasis on discussing two such things. So, let us first point out one very different key difference between the rotational motion and the translational motion. Now, we know that if we take a look at the displacement of an object, translational motion, let us say I can go along any path.

So, for example, I can go from here to there first. Let us say this is along the x direction, and this is along the y direction. I can go first along x direction and then take in the y direction. Let us say in the y direction, or I can first go along the y direction and then take the x direction. So, we both in either way, either route we reach at the same final position.

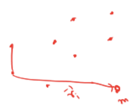
This is no longer true in the case of rotation. For example, suppose I take an object like this and let us say this is my x axis and this is my y axis, and I want to rotate this object along the x axis first and then the y axis. So, then I rotate along the x axis first and then the y axis. So, this is now the orientation of this particular object. Now if I do the opposite, I want to rotate the first along y axis by an angle 90 degrees.

Same angle of 90 degree as in the previous case, and then along y, about the y x axis, by the angle of 90 degree. Then I end up with the object with orientation. So, look, this orientation is very different from the orientation we got earlier, when we first rotated this object by an angle about x axis, first by an angle 90 degree and then about the y axis. So, this shows that finite rotation does not commute, just as finite rotation does not commute. So there is no vector addition for too-finite rotation.

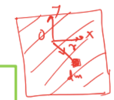
Finite rotation means rotating by about an axis y and a finite angle. So, this is the first kind of difference between translation and rotation. So this is kind of slightly counterintuitive. So in rotation, we can only talk about infinitesimal rotation, which is a kind of commute and we can define vectors out of that. So, when we talk about rotation, the key quantity is not the displacement but the rate of rotation.

So rotation by an infinitesimal amount. So we are going to discuss it in detail later. So here, I would like to point out the learning objectives for this part of this module. So first, we are going to consider the extended object and we want to define two key properties of the extended object that define their mass distribution. So the first property is the center of mass.

Center of mass: definition



$$\mathbf{R}_{CM} = \frac{\sum \mathbf{r}_i m_i}{M}, \quad \text{or} \quad \mathbf{R}_{CM} = \frac{\int \mathbf{r}\, dm}{M}$$



Problem type: Find center of mass of objects of simple shapes, i.e. how to write *dm*

So how do you calculate the center of mass position of an extended object? The second property is about something called moment of inertia which is required, which is the analog of mass but something more complicated and is a fundamental quantity to describe rotational motion. Then we will sort of look at the basic concepts and laws that are required to describe rotational motion and then we shall apply our systematic modeling approach to sort of analyze problems with rotational motion through several examples. First, let me give you some examples of extended objects. So here are, like, different examples which are different dimensions. So this one is an object, which is a line.

So this is a mass distribution distributed over a line, like a wire. So this is an example of a one-dimensional mass distribution. Then this one is an example of a set of gear. So if you assume that these gears are very thin, then, kind of, if you ignore the thickness of the gear, then this becomes a planar object. So this mass distribution is planar.

So this is an example of a two-dimensional mass distribution. Now if you include the thickness of such gears, you get a thin plate. This is, we can sort of think of it as something like a pseudo two dimensional mass distribution. Finally, we can get a fully three-dimensional object and a fully three-dimensional mass distribution. Now, from an engineering point of view, most of the engineering machines are kind of made up of different parts.

So, this is an example of something called composite objects. Which is made of different parts and in those cases, so these are like in the practical cases, Which are more important from an engineering point of view is this kind of composite object. Which are made up of different parts. Now, if you look at these examples, so each of the examples we are talking about objects with very simple shapes, like this is a line, this is a circular mass distribution, This is somewhat irregular mass distribution; this is a cubic mass distribution and In this case, this is an example of a composite object with a sphere-like combination of spheres and lines. Now, an interesting consequence of having finite size and shape, which I would like to point out.

So, if you look at around you and if you look at for example, where there are, you will find that These extended objects are broadly of two different varieties. The first one is what you call a compact object. A compact object like this cat or this ball, so in this case, they more or less have. Let us say there are a lot of objects that has more or less uniform density. So, the distribution of mass of that object and the distribution of volume of that object are kind of roughly proportional to each other.

So, the mass is proportional to the volume. So, if you take a bigger cat, its mass will be bigger, so if the volume of the cat goes increases by a factor of 2, so you expect the mass to also increase by a factor of 2. Now, there are another kinds of objects; the most prominent example is the tree. So, if you look at a tree such as this banyan tree in this picture. Most of the mass is actually concentrated in the trunk, which occupies a small volume.

Most of the volume is due to the surface area of the tree and the surface area of the tree is enormously large because of the leaves. So most of the surface area and volume of the tree come from the leaves, but the leaves have mass. Which are kind of negligible compared to the trunks. So in this case, this is an example of a shape or distribution in which the volume distribution and mass distribution are separated.

So it is for the trees. It is no longer true that If you increase the double the size of the volume. You can do double the size of the volume by increasing the number of leaves of the tree, but that is not going to add to the mass too much. On the other hand, you can double the mass by making the trunks thicker, but it is not going to add much volume to the tree. So, these kinds of objects are called fractal objects. Now there is a reason that if you look at animals, most of the animals follow these.

Like compact objects, and most of the trees follow this kind of fractal object. The reason is that animals are movable objects. So they need to conserve energy, so they need to minimize the amount of energy. They spend for movement, so that is why they are more compact. On the other hand, trees want to maximize their surface area to catch sunlight.

So that is why they have evolved to become fractal objects. So this is a side note. So now our basic goal is, so we will start our discussion by looking at these two properties of how to characterize this mass distribution. The first property is the center of mass. So center of mass, as you all know, so I must, and I am sure that you have studied center of mass in your high school.

So, if you have a collection of point masses, then there is a center, and each point mass has a mass mi and is located in space as a, let us say, this is a collection of point masses like this. This is, and let us say with respect to some arbitrary origin, the location of this point is Ri. Then the center of mass is given by the weighted sum of some weighted mass weighted by its location. So this numerator is a vector, M i times R i, R i is a vector. It is present as a bold and such that, and divided by the total mass of the system of point particles.

So that gives you the location of the center of mass. Now, instead of point particles, if you have a more continuous mass distribution, then this sum will be replaced by this integral. So, you have a small element of mass d M at, located at some, so this is a location of dm, with respect to my origin, so this is x, and this is y. And then let us say you have an object that is kind of square, and this is one element of the square, then the total mass and center of mass of the square will be given by the element times the location integral over all the squares divided by the total mass of the square. Now, so this is the mathematical definition.

So the question is, why do we need a center of mass? So take the example of a ball. So I am sure that all of you have seen some cricket match where a bowler delivers a ball. So the ball is an extended object. Why? Because the ball is a projectile. So, if you consider this ball as a projectile, then a projectile is a point mass.

Then, if you ignore all the forces except gravity, you expect that to follow a parabolic trajectory. If you include the effect of the drag force due to the air, drift force, etc. due to air, You may explain that the trajectory can deviate from the parabola so that the ball can swing. But the ball also spins about its own object on its own axis. So that motion is also crucial for the particular delivery.

And in order to explain that, you must assume that the ball is an extended object, the cricket ball. But now when the ball is moving, so it is spinning and moving. So, it is rotating about its own axis, and it is moving in space. So, it follows each point of the ball; if you track each point, one point. Let us say you put a dust particle on the surface of the ball and if you look at the trajectory of the dust particle as it moves through the air is a complicated trajectory.

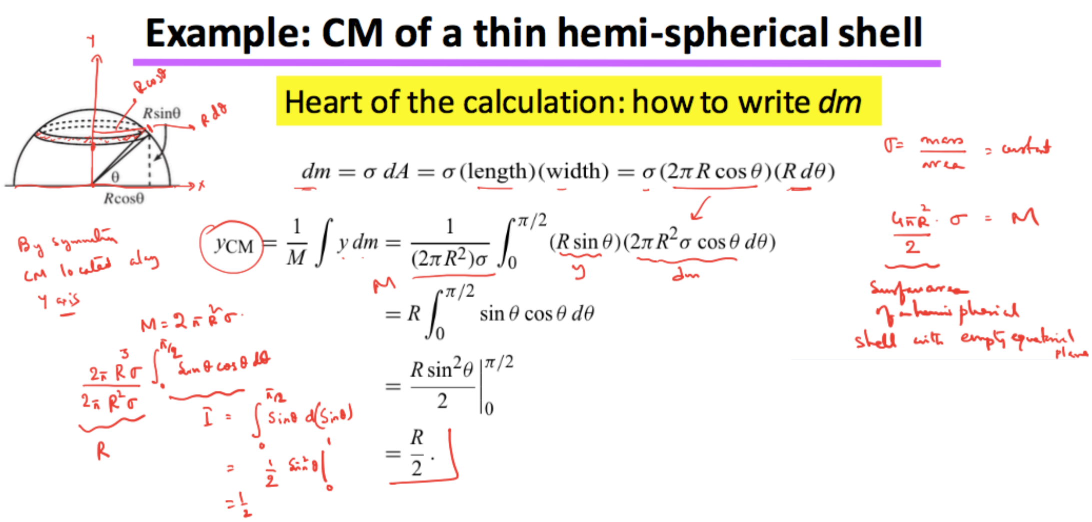
But on the whole, something in the ball is still moving in a simple, almost parabola-like trajectory. So this something, so this one, this is the representative point of that sort of represents the ball as a whole. So this is the center of mass. So you need that, so to a mass distribution, something that is extended in space, To give a first approximation, you can replace it with a point mass to describe its properties. And then you have to, the question will come that where you want to put this point mass, location of the point mass, and that location is given by the center of mass position.

And as you know that when you, when we consider the gravitational force. So, the gravitational force will act on each and every element of an object, but the, if you look at the total force on the full object, so that will still act through the center of mass. In this case, we are going to consider a problem of the following type: So we are basically going to, the problem that we are going to discuss is given an object that if you given the shape of an object, then what is it? Find the center of mass position. And in this case, we shall see that the essential problem, mathematical problem. How to describe a single, so we are going to do it in two steps.

So we are going to describe a single mass element dm, and then we do this integral to calculate the center of mass. So here are a few examples of objects of simple shape. So let us take a circle. So this is a ring, and here, in this case, the mass is distributed on the perimeter. So on the perimeter, it is shown as this blue line along a circle.

So this is an example as before; it is a 1D mass distribution. Now, these are kind of common examples from which it is relatively easy to compute the center of mass. So you can take a full circle or it could be a half circle or it could be a quarter of a circle or an arc of a circle, a part of a circle. And then the 2D version is that if the circle is filled instead of a ring, if you have a plate or a disk, so you can have a mass distributed throughout a filled circle or a half circle, or a quarter circle, or a part of a slice of a circle. So these are objects of simple shape in 1D and 2D for which we can calculate the center of mass easily.

Now if you take a circle and then rotate it about in three dimensions in the third direction, then you get a 3D object of revolution. Now, when you go from 2D to 3D, let us say if you rotate a circle about any of its diameter, Then you generate a solid sphere, as shown in this figure. Now some variation of a solid sphere can be a spherical shell or a hollow, empty spherical shell. So the mass is concentrated only on a thin shell but nothing is inside. the inside of the sphere is empty or it could be a half sphere, hemisphere or it could be a part of a sphere.



The other way to generate a 3D object from a two-dimensional circle is to consider the z axis, which is perpendicular, so this is x, this is y and this is z, which is perpendicular to the screen and then translate this circle along this z axis, then what you generate is a solid cylinder. And again, there could be variations of this kind of object, which could be a hollow cylindrical shell or a part of a cylinder, a cut cylinder or a part of a cylinder. So these are like objects; there are various common objects that are kind of useful to describe different practical engineering machines. Another example is a triangle, so another common shape in 2D is a triangle. So you can take a mass which is distributed in a field triangle or a part of a triangle in 2D and then what will be the centre of mass location and then from the triangle.

You can generate a three-dimensional object of revolution, which is a cone or it could be a conical shell, part of a corner, etc. So these are the various kinds of common shapes for which the problem is to find them. Given these objects or different shapes, these are some examples of the shapes. Find the centre of mass. Now I will take an example to sort of give you like, how to find out the centre of mass.

So consider that a thin hemispherical shell, so this is a hemisphere of radius r, and this is a shell, so the inside is empty. So, the mass is distributed over the surface of the shell. So what is the centre of mass location? Now, by symmetry we can see that the x, so if this is my x axis and this is my y axis, By symmetry, we can see that the mass distribution is uniform, so the centre of mass must be located along the y axis. So the question is that what will be the height? At what height from this x-z plane? The equatorial plane of the sphere will be the centre of mass? So for that, let us consider a ring, a ring as shown here, so this ring is at an angle theta with the x axis. So this is the elevation of the ring, and it has an area, which is given by the length times the width.

So the ring perimeter is this length, which is 2 pi r cos theta because the radius of this ring is r cos theta. And then the width of the ring is this arc length, and this arc length is given by r times d theta. So, the area element of this mass, this strip of ring, will be sigma, so sigma is the density, so the mass per unit area is, so we are going to assume this is constant. So it is uniform, so this is given by this particular expression. And then, by definition, the y location of the centre of mass coordinate is given by the y times dm, so the location of any y coordinate of any point of this ring that is the elevation of this ring is given by r sin theta, which is clear from the picture.

So, this is y, and this is dm, which is given by 2 pi r square sigma cos theta d theta. And then the total mass m, so we can sort of calculate easily that the, If you take a sphere, then the total surface area of the sphere is 4 pi r square and time sigma. That should be the total mass for a sphere, but we have a hemispherical shell. So the surface area and this equatorial plane are empty; this is our assumption, so this is the surface area of a hemispherical shell with empty equatorial plane, so this is, so this is m, so this is m. So, this factor is the integral and then this is at a particular length, theta, so if we sort of move this ring up and down from all to cover all possible values of theta, then we have to integrate it from 0 to pi by 2, which covers all possible values of theta.

So, after simplification, we see that this integral is given by, so m is 2 pi r square sigma, so what we have on the numerator is 2 pi r cube times 0 to pi by 2 sin theta cos theta d theta and times sigma, and in the numerator we have 2 pi r square sigma. Now, this is integral, so let us call that I, So, you can see that this is sin theta times d of sin theta because 0 to pi by 2 sin theta is 0. Because d of sin theta, so if you put sin theta equal to x. Then this is just x dx, so this is half sin square theta and x theta is equal to 0, sin theta is 0. Theta is equal to pi by 2; sin theta is 1, it goes from, so this is just half.

So, what we get is that, after calculating cancellation, only r remains. So, this is r times half, so you get this location to be at the middle, so this is probably something. We can guess by symmetry, so the location of the centre of mass will be of a hemispherical surface here will be at the middle along the y axis. So, we shall consider more interesting examples of centers of mass in the next lecture. Thank you.