Course Name: Newtonian Mechanics With Examples
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Week 06
Lecture - 33

Welcome to the seventh week of this course on Newtonian mechanics with examples. We shall start this week by taking a few more examples of collision problems. In the last couple of lectures, we took some examples of collisions from our day-to-day life. However, I want to say that collision, which is also called scattering, is a very important problem, which in the last century was very instrumental in making fundamental progress in physics. So because these collisions were used as a probe to study the structure of the matter. What is the matter made of? So I will take one or a couple of examples from today's lecture to sort of tell you something about this fascinating topic.

Our first example is something called Brownian motion. So this was discovered way back in 1827 by a botanist called Robert Brown. So, as you can see in this GIF file, what he observed was that pollen grains in water, If you look at them and observe them under a microscope, you will see they are not at rest, but They are moving something like that, as shown in this picture. So he observed, but he could not understand why they were not at rest but were moving.

After almost 80 years, Einstein proposed a theory in 1905. So he realized that these pollen grains, which are very small microscopic particles, So, you cannot see them through naked eye, but you have to look at their micrometer-scaled objects. So, you will have to see them under a microscope. So they are not at rest because there are collisions between these pollen grains and the molecules of the liquid in which it is. So here is an animation of the same.

So this is an animation, not a real experiment. So imagine this yellow sphere, which is a large object, so this represents pollen grain and These small black dots are the molecules or particles of the fluid. So these are small particles, so we cannot observe them directly with our naked eye, not even under a microscope. So the only thing we observe is the trajectory of the movement of the sphere. So, Einstein imagined that this is a picture where these fluid molecules are continuously hitting this yellow sphere and that is why we see this yellow sphere doing this kind of erratic apparently random movement in the fluid.

So, this apparent random movement is very important in physics, so this is an important phenomena. Which is called diffusion, and this is technically called Brownian motion. So, after Einstein proposed that this is due to the atoms and collision with the atoms and molecules of the fluid, John Pera did an experiment in which he actually proved that this is indeed due to the atoms. How did he do that? Because he sort of proposed Einstein's theory, he measured the Avogadro number. So for this groundbreaking experiment, he won a Nobel Prize in physics.

The significance of Einstein's theory and Pera's experiments is that this is the first direct experimental evidence that atoms and molecules exist. Even though it was proposed long before by chemists, physicists in the 19th century did not sort of believe that atoms really exist. They thought that this was just a useful mental picture to explain the chemist's experiments. But when Einstein and Pera proposed their explanation of Brownian motion, Physicists were forced to accept that matter is not continuous. In fact, it is made up of atoms and molecules.

So in this context, so I will refer to you a particularly interesting website that I discovered when I preparing for this lectures, so which sort of gives you a more background about this fascinating topic of Brownian motion. Today instead, we are sort of look at this collision between this sphere and the particles and we are going to take an example in which we are going to look at and We are going to derive the expression for the drag force on a sphere. So remember in earlier week we derived an expression for the drag force by dimensional analysis. Today, our first goal is to derive an expression for this drag force on a sphere which is moving inside a viscous liquid from the collision picture which is the more physical picture. So the picture is the following:

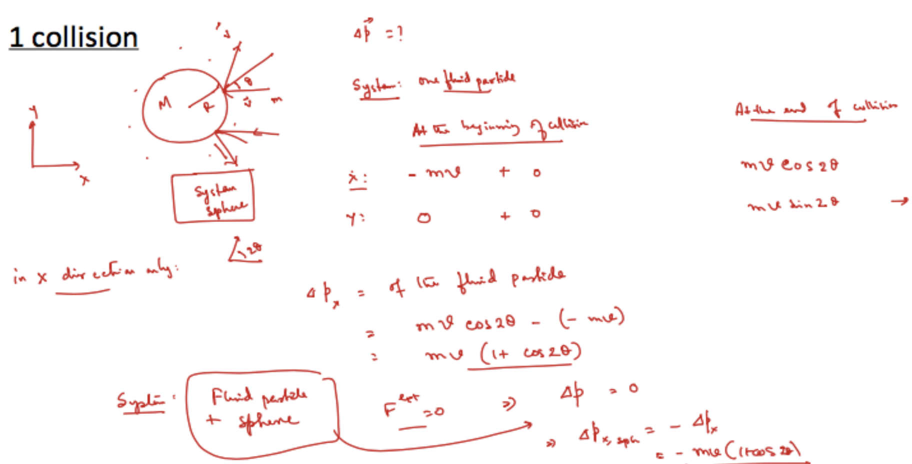
Example 28: Drag force on a sphere

A sphere of mass M and radius R moves with speed V through a region of space that contains particles of mass m that are at rest. There are n of these particles per unit volume. Assume $m << M$, and assume that the particles do not interact with each other. What is the drag force on the sphere?

So suppose there is a sphere of mass m and radius r which is moving with a speed v. Through a region of space that contains particles of mass m that are at rest. So these particles of mass m are basically fluid particles. There are n of the particles per unit volume. So the number density of this fluid particles is small n and assume this is crucial.

That small m is very, very smaller than this sphere. So the sphere is much bigger than the fluid molecules. So to give you some length scale, imagine that, now we know that the atoms and molecules are of the size of angstrom. So, they are like nanometer objects, whereas this sphere is at least 1000 times bigger. So they are like micrometer-sized objects.

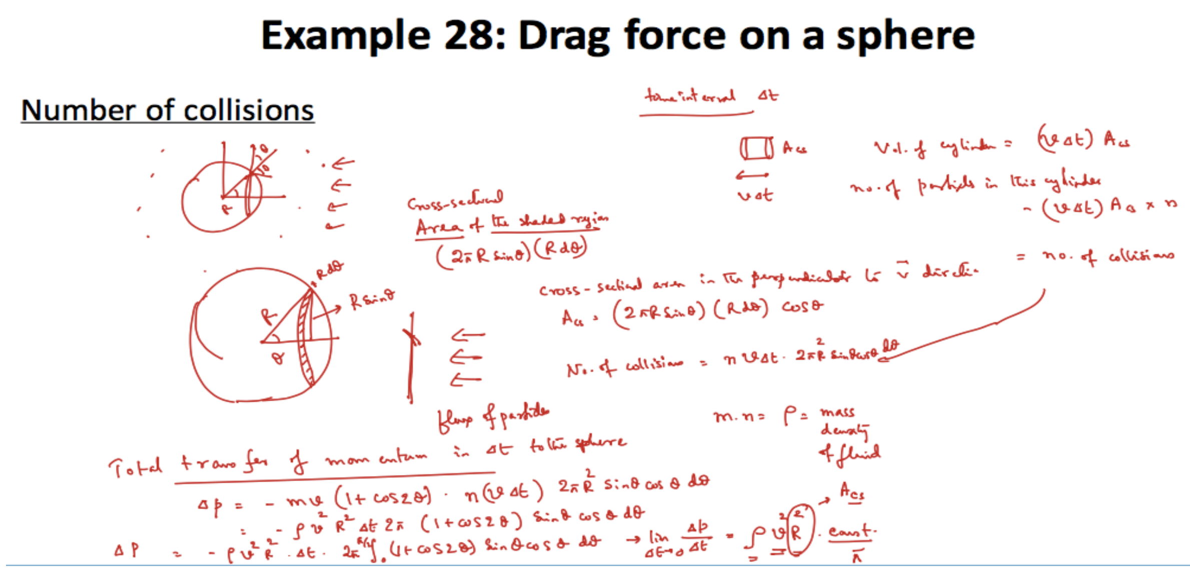
Example 28: Drag force on a sphere



And assume that the particles do not interact with each other. So, this is to simplify the calculation. So, then, the problem is to derive an expression for the drag force. So, we are going to divide this into two steps. So our physical idea is the following.

So we have, imagine this is a sphere of radius r and this sphere is immersed in this fluid and let us say that this is a fluid particle of mass m. So this has a capital M; this has a mass small m. So this particle hits the sphere, and then it rebounds in this direction suppose. Now this angle is called theta. So, for simplicity, we are going to assume this is our x direction and this is our y direction.

So this is v1, and this is after collision v2. So what happens is that? This m, when it hits the fluid particle, when it hits the sphere. It transfers some of the momentum to the sphere, and then it gets rebounded. So now this sphere is moving with a velocity v. So, we can also imagine in the following way that it is the, let us say, sphere at rest and then if you are sitting on the sphere, you will see that this is fluid particle is approaching the sphere with a velocity v in the opposite direction.



So, this is going to be our assumption. So, this is a trick to make the calculation simple. And our first question is: how much is the change in momentum during one collision? So let us define. So our system is this sphere, and then this fluid particle is going to give a force on it. Now, from this picture, you can see that the fluid particles can hit the sphere at any position.

So then you are going to see that first thing is that by symmetry, there is no change net momentum transfer in the y direction. Because if it hits from here, let us say it hits from here and gets rebound in this direction. Then this is a symmetrically opposite case as in the top picture. So, then the y component of the momentum of the fluid particle in this collision and versus this collision they are equal and opposite, so they will cancel. So all you need to consider is a net change in momentum in x direction only.

So let us first look at the change in momentum of the particle. So let us, for the time being, say our system is temporarily one fluid particle. So, at the beginning of the collision and at the end of collision, the beginning of the collision, this is incoming with a speed the x component is mvx; this is totally x, and we are going to assume, so we are sitting on the sphere. So, we are going to assume that the sphere's initial momentum was 0. At the end of the collision, this rebounded fluid rebounds in this direction.

So, the mv, then the x component, so this angle with the x axis is 2 theta, so then of the rebound velocity, so the x component is cos of 2 theta. So, the change. And this is in the minus direction, and this is in the plus direction. We can write down the y component, so the y component will be minus and we can simply write it as x because the x component is the initial velocity of the fluid particle, which is completely in the x direction. So similarly in the y direction, the initial was 0 and the final is mv sin 2 theta.

But we are not going to consider this because they are going to cancel when we sum over all the collisions. So, let us look at the change in momentum of the fluid particle. Change in momentum in the x direction, so this is mv cos 2 theta minus minus mv, so this is mv 1 plus cos 2 theta. Now let us say that if we look at the fluid particle plus this sphere as our system, then there is no other external force, so we are going to ignore any gravity and anyway, gravity is not in the horizontal direction, so there is no external force on this particular system. That means the delta p of this system in this time interval must be 0, That means a change, so this is the change in the momentum of the fluid particle.

So, that means the change in momentum of the sphere must be opposite of this change in momentum of the fluid particle, which is, so this is the transfer of momentum to the sphere by the fluid particle as a result of the collision. And this is what this sphere will experience as a force, because, as we say that this is the impulse and if you divide by the time interval you get the average force. But before we estimate the average force, we need to estimate how many collisions are taking place. So here the picture is the following: so far we have considered just one collision. Now, there is a lot of particles inside this fluid and they are hitting the sphere from all sides, so the sphere is moving to the right with a speed v, which means if you are sitting on a sphere, the particles are approaching the sphere with some speed v.
So, what is the net number of particles that are approaching? So, let us say that this angle is theta. Let us say that what is the surface? So we need to calculate this surface area. So if you take it, let us make a bigger picture. So the number of particles that are hitting this ring, so imagine a ring on the sphere, So sphere is a 3D object, and now on this ring, all the particles are hitting the sphere with an angle theta, as shown in this particular figure. So they all have the same angle, theta.

So how much is this area of the ring? So, the total number of collisions is the total number of particles hitting this area. So how much is this area? So this area, so the, it is kind of a making a ring with, so this is r, then with this radius is r sin theta. So we have a perimeter, which is 2 pi r sin theta times the width, which is r d theta. So this is the area of the shaded region. However, what we need is a slanted area.

What we need is a projection in the vertical direction. Because this is what the particle will see as the area, these particles consist of a flux. So there is a flux of particles. So how much it will hit the sphere is determined by the cross-sectional area in the perpendicular direction.

So this is the cross-sectional area. And in the cross-sectional area in the perpendicular direction, perpendicular direction to v direction, let us call it A c s will be, you can convince yourself that this will be the above area multiplied by the projection in this direction. Now imagine that there are n particles in some volume. So now you are looking at a time interval delta t. In this time interval delta t, the particles move for, they are moving with, all the particles are moving with speed v. So they will be contained in a cylinder of length v delta t and a cross-sectional area Acs.

And so then the volume of the cylinder is equal to v times delta t into A c s and the number of particles in this cylinder is equal to v times delta t times the cross-sectional area into n, where n is the number density. So, all these particles are going to collide with this sphere in this particular time interval, delta t. So this is the number of collisions. So if I write it here, this number of collisions is equal to n v delta t times 2 pi r square sin theta cos theta d theta. Now we have estimated that one collision, the net transfer of momentum on the sphere is minus mv times 1 plus cos 2 theta.

So total transfer of momentum in delta t time interval to the sphere. This is the change in momentum of the sphere, which is given by the transfer of momentum in one collision of this particular type into n v delta t times 2 pi r square sin theta cos theta d theta. So then this m times n is the mass density, so this is the mass density of fluid. So let us call it delta p. So this is equal to rho, we have v square, we have r square; and We have 2 pi times delta t times 2 pi times 1 plus cos 2 theta sin theta cos theta d theta.

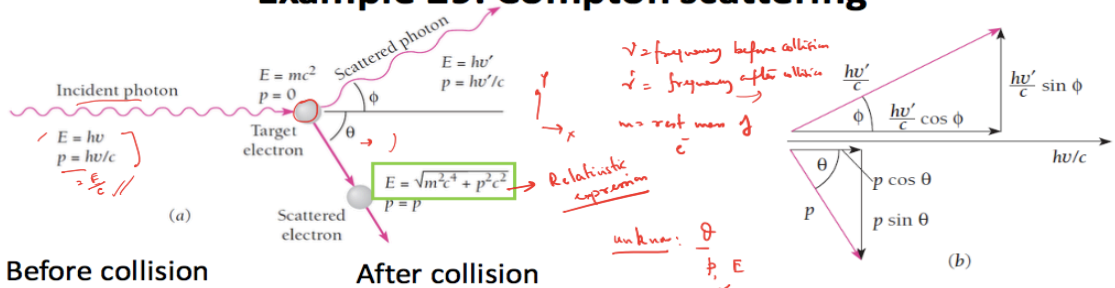
So we are just rearranging the term. Then if I, now this theta can vary from 0 to pi by 2, so then we have to integrate over all possible values of theta, then we get. Now note that this integral, we do not need to know what the value of the integral is to get the force. Because now the force is given by the definition that If we now take the interval going to 0, which is very small, then the rate of change of momentum is given by this force. So then we get times some constants. So at this point, we do not need to know what the constant is; it is, in fact, If you calculate it, you will get that the constant will turn out to be pi.

But the important point is that we get this dependence of v and we get back this v square dependence is sort of derived based on dimensional analysis and we get some r square, So, it is some length scale times square, which represent the dimension of the cross-sectional area. And this drag force, so I am only considering the magnitude, is again proportional to the density of the fluid. So, this was the conclusion before, when we based it on the dimensional analysis. Now we arrive at the same conclusion based on the actual physical picture of collision between the fluids on the sphere.

So this is how the drag force originates. Now one more thing: we have considered a sphere, but it need not be a sphere. It can be of any other shape and all it will change is the value of this integral, so The details will be changed, but this dependence on v. The fact that this force will be proportional to the density of the fluid, proportional to the velocity square of the object and proportional to the cross-sectional area, so some linear dimension of the object square, these facts will remain the same. In the next example, we are going to consider collisions at a much smaller scale, at atomic and subatomic length scales. So let me remind you that you have read about these examples in other courses, such as modern physics or quantum mechanics.

So for example, here we mention this famous alpha particle experiment by Rutherford. Which shows that most of the matter is actually empty space. So, from a mechanics point of view, this is an experiment in which there is a collision between the alpha particles and the nucleus of the atoms in the gold plate. So, this is called, in physics terms, the example of scattering, which is another word for collision. Now we are going to take another example, which is called Compton scattering.

Example 29: Compton scattering



Before collision **After collision**

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\phi + p\cos\theta$$

$$0 = \frac{h\nu'}{c}\sin\phi - p\sin\theta$$

$$h\nu + mc^2 = h\nu' + E$$

So, which is another sort of extremely fundamental experiment in physics. So you have also seen, and I am sure you have seen some of you who has taken courses in modern physics or quantum mechanics, you have come across this example in those courses. So, instead of alpha particle, you send X-ray beams, which are also made of light particles, or photons, and there is a scattering of those photons by the atoms in the crystalline matter. So, this experiment proved that the photons of light not only carry energy, but they also carry momentum. So, here, I show the schematic of the setup of the original experiment.

So, this example we are going to, this experiment we are going to consider a little bit in more detail. Now this is a course on Newtonian mechanics, but this example, I am going to differ. And we are going to take some steps outside of Newtonian mechanics in two ways. First of all, these atoms and the photons are no longer classical particles that obey Newton's laws of motion. In order to analyze this example, we really need quantum mechanics; they are quantum particles.

The second thing is that the photons move at the speed of light. So in our course so far, all the velocities are much smaller than the speed of light. This is no longer true here. So, here we must use the relativistic expressions for momentum. So, my reason for showing this example is to show you that the conservation of momentum and energy, so this has a much broader scope of applicability beyond Newtonian mechanics.

You do not need the particles to follow Newton's laws of motion for the conservation laws to be valid. Here is an example where the particles follow quantum laws of motion and relativistic speeds and still the conservation laws are valid. So in this case, we can use the same setup of our strategy that we are using so far to analyze this example. So, I am going to quickly go through this example. So, at the very basic level, this is a simple two-body collision picture.

So, this is an incident photon of X-ray, and this is a target electron inside an atom inside the matter. So, when the photon is carried, it has some energy E, which is equal to Planck's constant time is its frequency, so this is by the De Broglie relation and it has momentum. We know that the energy-momentum relation is E by C, so the momentum is E by C. So this is the relativistic expression for the momentum of a particle. So note that the concept of momentum is now generalized to include photons, which are massless particles.

So, we cannot use momentum to be mass times velocity because photons carry no mass. So, it carries energy, and you can write the momentum in terms of the energy. In the second part of this relativistic, it hits a target electron. As a result, it scatters in a different direction, and this scattered photon. So you see that scattered photons wavelength is now bigger than the incident photon, which means its frequency, which is inversely proportional to wavelength lambda is now also changed.

So after collision, let us say the frequency of the photon is nu prime and It goes to a direction which makes an angle phi with the original direction and the electron also moves in a different direction, making an angle theta such that these two body systems of incident photon and the target electron keep the photon plus electron, the total momentum of this system remains conserved. So before collision, so if we write down the expression for momentum in the x direction, So, this is the momentum of the photon which was completely in the x direction and We are going to assume that the electron was at rest. So, this is the momentum of the electron before collision. After collision, the photon moves in a different direction, so it is a different photon now with a different frequency. It moves in this direction, so its x component is h nu prime divided by c times cosine of phi and the component of the electron's momentum in x direction, so this is my x direction.

This is my y direction, so this is x, and this is y, given by p cos theta. And similarly, before collision, there was no momentum of either the 90th of photon nor the electron has any momentum in the y direction, so before collision, the net momentum is 0. After collision, the y component of the momentum of the photon is h nu prime by c. Which comes from this relation, sine phi minus p sine theta. So, they sort of must cancel each other, so this represents the conservation of momentum.

So, this is also what I said just now; it is also explained vectorially in this picture. And the energy, Now look at the energy, so before collision, The energy of the photon was h times nu, nu is the frequency, so let me write it down. Nu prime is the frequency after collision. Now the photon and the electron are at rest, but here is the relativistic formula that still it has some energy, which is given by m c square, E is equal to m c square, Where m is the rest, called the rest mass of the electron. So this is a concept that, in relativistic concept, is that mass is a function of velocity speed.

So, that we cannot go into detail, so you have to start to assume that This is the relativistic expression for energy. And after collision, the energy of the photon is h nu prime and the energy of the electron is E. So, this is the kinetic energy of the electron, but this is no longer p square by 2m. But it is given by this relativistic expression. So conceptually, there are lot of new things in this example, however, in terms of mathematics.

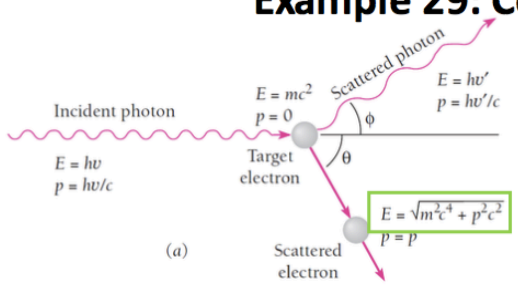
The rest of the analysis is pretty straight forward and simple, so you have these equations and then you have to solve these equations and calculate. So what are the unknowns here? So, the unknowns, so the known, are this: the frequency of the incident photon. During the experiment, we know what the frequency of the x-rays is that we are sending to hit our target electron in this crystal. And we can also measure the angle phi directly in the experiment by looking at the direction in which this scattered x-ray is collected. But theta is unknown, so theta is unknown; the other unknown is the momentum of the electron after collision, so which is the energy of the electron after collision, because We do not know what happens to the electron inside; we cannot directly track the properties of the electron.

So, what we can do is eliminate, so note that there are three. There is another unknown, which is the frequency of the scattered x-ray. Now you have three unknowns and three equations, so you can exactly solve them and get an expression for the frequency. So, this is something that I am not going to show you in detail. But I will leave it as a take-home exercise to solve this set of three equations and calculate the frequency after collision. So I will outline the strategy that you should use, and I can give you a hint.

So what you do is that you try to eliminate theta, so in order to eliminate theta, what you can do is. Let us say if I take the first equation and multiply both side by C, what we get is P cos theta is equal to, so let us say P C cos theta is equal to h nu minus h nu prime cos phi.

And then on the other hand, the P sin and P C sin theta are equal to h nu prime sin phi. So, you can use this identity, cos square theta plus sin square theta equal to 1, to eliminate theta from these two equations. And then you eliminate E; after that, in the next step, you eliminate E from the third, using the third equation.

Example 29: Compton scattering



$c p \cos\theta = h\nu - h\nu' \cos\phi$

$c p \sin\theta = h\nu' \sin\phi$

c = speed of light

$\lambda' = \frac{c}{\nu'}$

Outline of derivation:

1. Eliminate θ
2. Eliminate E

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\varphi)$$

→ derive Compton formula

Then finally, you arrive at a expression, so which is shown here in terms of the wavelength instead of the frequency, so this wavelength after of the scattered photon is given by C by nu prime, where C is the speed of light. So, work through the algebra and show that the change in the wavelength of this x-ray due to the collision with the electron is given by this particular expression. Which is known as Compton formula. So, there is a typo in this equation, so this should be a cos phi. So, this is the Compton formula for which to do this experiment and to show that the photons and this experiment proves that the photons carry momentum.

So, I will let you work through this algebra and derive this very famous problem. So, this is an example to sort of get a flavor of how collisions are very fundamental to the progress of physics. This is an example of light-matter interaction, where matter is interacting with light. So, this is a topic of great fundamental interest in frontier cutting-edge research in physics even today.

So with this, we are sort of completing our discussion of the collision problem. So, next, we are going to look at another new topic, which is the translation and rotation of rigid bodies. Thank you.