Newtonian Mechanics With Examples

Shiladitya Sengupta
Department of Physics
IIT Roorkee
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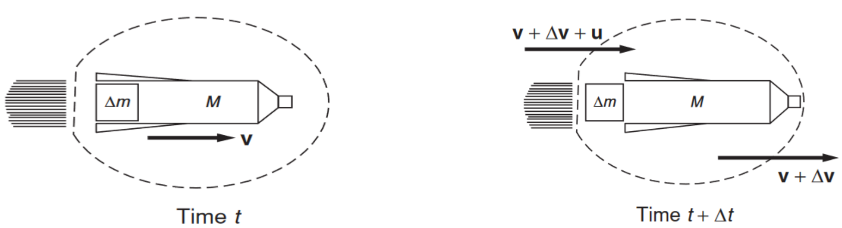
Lecture-31

In the last few lectures, we were discussing various kinds of projectile motions. So, you are doing this sort of as an example, as a basic application of the different laws that we have discussed so far, that is, Newton's laws, the conservation of energy and conservation of momentum. So, in particular in the last lecture we talked about the thrust force and it is, so, this force is usually responsible for the rocket motion. So today the goal is to discuss this in more details through an example. So in this example we have a rocket. So, imagine a rocket which is moving in deep space.

So, in deep space, there is no fluid surrounding the rocket, so, there is nothing to help the rocket to accelerate. And we are, deep space means which is far away from the earth's or sun's gravitational field, so there are no, we are going to assume for simplicity that there are no external force acting on the rocket. So now the rocket will move it at a constant velocity but now suppose we want the rocket to accelerate, so then how can the rocket accelerate? So, this is where the thrust force coming into the picture. So, imagine that at, so what happens is that the mechanism is the following.

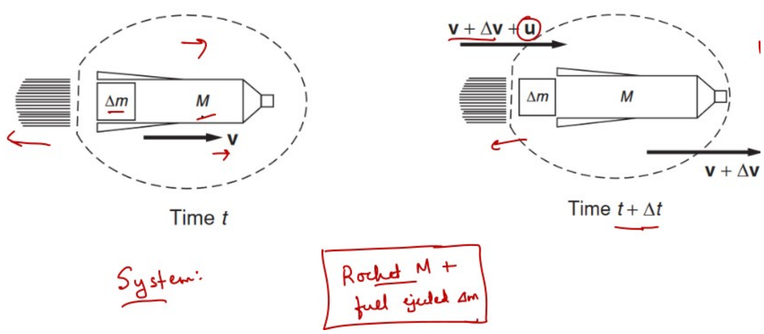
So, imagine that, look at the rocket for a time interval of Δt. So, what happens is that the rocket has, let us say its mass is M at time t, and part of the mass is the fuel. So this fuel, let us say its mass is Δm, and the rocket is going to eject this amount of fuel in the time interval Δt, out. So, it is going to exert a force on this fuel to eject it in this direction. And then the, by Newton's third law, the fuel is going to push the rocket in the opposite direction.



And that is how, so this is the thrust force, and that is how the rocket can accelerate. Now this problem is, so this actually, remember that when we discussed in week 2 about Newton's laws of motion, we said that there are, the most general version of the Newton's law is called, so where you, instead of writing F=ma or F=dP/dt, we write as a change, look at the change in momentum and equate it to the impulse. So, that version is really useful in this case because this is a variable mass problem. So, the rocket's mass is changing with time. So in this case, it is confusing to sort of look at what is the momentum.

So, we have to be careful. So let us analyze this problem. So for the sake of concreteness, we have, let us say, we are going to write at the start of interval and at the end of interval. So, here, what is the mass of the rocket? So, the mass of the rocket is the mass of the rest of the rocket plus the mass of the fuel that is ejected. Then, and note that this mass is changing with time.

Next, what is the velocity of the rocket? So, the velocity, now here we have to be careful. So, this velocity is with respect to, let us say, a distant observer or, let us say, the ground station. And let us imagine that the rocket was moving with a velocity of v to the right. What is the momentum of this system? So, okay, let me define the system. So, our system, in this case, is the rest of the rocket plus the fuel ejected.



So, this is going to be our system. Now everything else is surrounding. So the rocket, remember in deep space, so we are going to assume for the time being that there is no other external force acting on this system, only internal force, the rocket and the fuel exerting on each other. So this is the situation. So, now we can write down the momentum of the system at time t. So, the momentum is simple.

At the start of interval

Mass of rocket $= M + \Delta m$

Velocity of rocket w.r.t ground $= \vec{v}$

$$\vec{P}(t) = (M + \Delta m)\vec{v}$$

At the end of interval

Mass of rocket $= M$

Velocity of rocket w.r.t ground $= \vec{v} + \Delta\vec{v}$

mass of ejected fuel $= \Delta m$

velocity of the ejected fuel w.r.t. ground $= \vec{v} + \Delta\vec{v} + \vec{u}$

$$\vec{P}(t+\Delta t) = M(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \Delta\vec{v} + \vec{u})$$

Now, let us write down these corresponding quantities at the end of the interval. So, at the end of the interval, note that the fuel, so this is the picture at the end of the interval. So here, note that the fuel is now separated from the rest of the rocket. So, these are like really different parts of the same system.

So the mass of rocket is now M. So it has reduced. Now the obviously, because the fuel gives a push to the rocket to the right, its velocity has changed. So I am going to write, assume that it is changed by an amount ∆v and the goal of the problem is to calculate how much is the change ∆v. So this is with respect to ground.

Now what is the mass of the fuel? So the mass of the fuel, ejected fuel remains the same. So, this is ∆m. Now what is the velocity of the ejected fuel? Now here you have to be careful, and this is where a lot of confusion comes. So, let us assume that u is the velocity of the fuel with respect to the rocket. So, with respect, if you sit on the rocket, then you will see that the fuel is ejecting towards right within velocity, towards left within velocity u.

Then, from the ground, by applying vector laws of addition, the velocity of t∆vhe fuel with respect to the ground should be the velocity of the rocket with respect to ground plus the velocity of the fuel with respect to rocket. So, then this is the velocity of the rocket with respect to ground, and this is the velocity of the fuel with respect to rocket. So, the velocity with respect to ground will be this much. So, then we can write down the momentum of the full system at the end of the interval at t +∆t and and P at the beginning of interval. Now we can easily, so this way, we can easily calculate the change in momentum in an interval ∆P. So, we just have to take the difference, and when I take the difference, we see that

$$\vec{P}(t+\Delta t) = M(\bar{v}+\Delta\bar{v}) + \Delta m(\bar{v}+\Delta\bar{v}+\bar{u})$$

$$\bar{P}(t) = (M+\Delta m)\bar{v}$$

$$M + m = \text{constant} \Rightarrow \frac{dm}{dt} \rightarrow -\frac{dM}{dt}$$

$$\Delta\bar{P} = M\Delta\bar{v} + \Delta m(\Delta\bar{v} + \bar{u})$$

$$\frac{\Delta\bar{P}}{\Delta t} = M\frac{\Delta\bar{v}}{\Delta t} + \frac{\Delta m}{\Delta t}(\cancel{\Delta\bar{v}} + \vec{u})$$

$$\lim_{\Delta t\to 0} \frac{d\bar{P}}{dt} = M\frac{d\bar{v}}{dt} + \frac{dm}{dt}\vec{u} \rightarrow \boxed{\frac{d\vec{P}}{dt} = M\frac{d\bar{v}}{dt} - \frac{dM}{dt}\vec{u} = \vec{F}^{ext}}$$

Now divide both sides by the time interval to get the rate of change of momentum. So, now if I take this time interval to be smaller and smaller, so basically infinitesimal, then in the limit when the time interval is very small, the change in ∆v is going to be smaller, but the ejection of the fuel which is determined by the mechanism, internal mechanism by which the rocket is pushing the fuel to the outward, that will not change. So, then in the limit of Δt going to 0, then the left-hand side will become the rate of change of momentum.

In the right-hand side, the first term similarly will become the change in the mass times the rate of change of velocity and this term we can drop, because this term is going to be negligible compared to u. Now note that we can further simplify. So this we can eliminate this m, the mass of the fuel, because what we are interested in usually or we keep track in usually is the mass of the rocket. So, the total mass is the mass of the rocket and the mass of the fuel, mass of the rest of the rocket and the mass of the fuel is constant.

That means any change, the rate of change of the fuel ejected from the rocket is going to be inverse negative of the rate of change of decrease in the mass of the rocket. So, then we can simplify this equation. So, now you see that this is not simply just a straight forward F=M**a** or F=d**P**/dt. So this is why we discussed this most general version of Newton's laws of motion at the beginning of this course, which is very useful in this kind of situation when you have different parts of the system and the mass is also changing with time. So now once we have this equation, then we can now sort of, so we can apply our momentum principle and then we can sort of, in general, it will be equal to the external force acting on the system.

Now we are looking at a situation in deep space which means the external force we are going to assume to be 0. Then our equation says that, now this equation once we derive some relation, one thing you should always do is to think that does it make sense. So, how do we look at it? So let us check the sign. So we expect that the as a result that because of the fuel ejecting, the rocket should accelerate, which means dv/dt should be positive. Now let us look at in this picture, so the sign of the second term.

$$\Rightarrow \underline{\bar{F}^{ext} = 0}$$

$$\boxed{\frac{d\bar{P}}{dt} = M\frac{d\bar{v}}{dt} - \frac{dM}{dt}\bar{u} = 0} \rightarrow (1)$$

$$\frac{d\bar{v}}{dt} = \frac{1}{M}\frac{dM}{dt}\vec{u} \rightarrow (2)$$

$$\frac{d\bar{v}}{dt} \downarrow \; >0, \quad \frac{dM}{dt} \downarrow \; <0, \quad \vec{u} \downarrow \; <0$$

$$\Rightarrow \; >0$$

Now in this picture, the fuel is going towards right, that means the sign of u, so note that it has a vector sign, so the sign is included in the symbol, u must be negative. And as a result of ejecting, ejection of the fuel, the rocket's mass, the rest of the mass of the rocket is also decreasing, which means dM/dt is negative and this is also negative, so that means that this must be positive. So, this makes sense because we, the rocket, so is trying to accelerate to the right. Now we can sort of look at, let us say, let us try to solve this equation and find how the mass changing with time.

Now, I am going to write equation 2 again. Let us try to integrate-

$$\frac{d\vec{v}}{dt} = \frac{1}{M}\frac{dM}{dt}\vec{u}$$

$$\int_{t_0}^{t_f} d\vec{v} = \vec{u}\int_{M_0}^{M_f} d\ln M$$

$$\vec{v}_f - \vec{v}_0 = \vec{u}\ln\frac{M_f}{M_0}$$

$$\boxed{\vec{v}_f = \vec{v}_0 - \vec{u}\ln\frac{M_0}{M_f}}$$

↳ does not depend on $\frac{dM}{dt}$

Now if I integrate from the beginning over some interval, let's say t_0 to t_f, let's say the mass in this integral, over this interval, that the beginning of, at $t=t_0$, the mass of the rocket was, including the fuel was M_0 and at $t=t_f$, the mass was M_f.

Now note that the rocket's mass is decreasing, so M_f is less than M_0, that means this log must be negative of that. So we can sort of acknowledge that.

So note an interesting point about this equation. So it says that the final velocity of the rocket at the end of this interval depends only on the mass at the end of the interval and the mass at the beginning of the interval. So it does not depend on the rate of change, that is at what rate the fuel is ejecting from the rocket. So the fuel can eject very, very slowly from the rocket or it can eject very rapidly from the rocket, but as long as the mass at the ratio of the mass is same, you get the same final velocity. So then a question naturally comes into mind is that, so when the rocket lifts off, takes off from the, from the earth, this is usually a very spectacular phenomenon.

So the, why the rocket launches are so spectacular? In other words, when the rocket launches, then a lot of fuel gets ejected in a very short amount of time and that is what creates these magnificent jets that we see during the takeoff. So, the reason behind this is the gravitational field. So, in the deep space, there are no external field acting on the rocket. However, when the rocket is taking off from earth, we cannot ignore earth's gravity.

So, just define the system. So this is our rocket and plus the fuel it carries. And then in this case, in the surrounding, there is this earth's gravity and this is exerting a force. So there is some exerting force and we are going to assume for simplicity that this is the only force which is not true in real life. So in actual rocket launches, you have to design in your, when you design the rocket launch, you have to take into consideration the fact that air is also, the rocket is passing through air. And also the earth is not a stationary object.

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext} = M\vec{g}$$

$$\frac{d\bar{P}}{dt} = \frac{M d\bar{v}}{dt} - \boxed{\bar{u}\frac{dM}{dt}} = -M\vec{g} \quad -③$$

Rocket + fuel (system)

↓

earth's gravity

$$\frac{d\bar{v}}{dt} = \bar{u}\frac{1}{M}\frac{dM}{dt} - \vec{g}$$

$$\int_{V(t_0)}^{V(t_f)} d\bar{v} = \bar{u}\int_{M_0}^{M_f} d\ln M - \vec{g}\int_{t_0}^{t_f} dt$$

$$\uparrow \quad \vec{V}_f - \vec{V}_0 = \bar{u}\ln\frac{M_f}{M_0} - \vec{g}(t_f - t_0)$$

$$\downarrow \quad \bar{V}_0 = 0 \quad \vec{V}_f = -\bar{u}\ln\frac{M_0}{M_f} - \vec{g}(t_f - t_0)$$

It is actually spinning about, it is rotating about its axis. So those things you have to take into consideration. But for this problem to we are going to just make it drastically simple. So we are going to assume that the only force on the, in the surrounding that is interacting with our system, so this is our system, is the mass or the weight, the earth's gravity. So in this case, so let us rewrite this equation.

In other words, we dP/dt was equal to Mg. Now let us solve this equation. Let us call it equation 3. So again we divide both sides by the mass, then we get the change in acceleration of the rocket is given by the u(1/M)(dM/dt) as before, but now there is an extra term which is the acceleration due to gravity.

So, during the few meters at the beginning, so we are now focusing on the beginning, few seconds where the rocket is still close to the ground. So we are going to assume that g is constant. So this is again a simplification. So then let us say if we want to know how much is the change in rocket's mass over a time interval t_0 to t_f, then it is going to be, so this is an easy integration.

So let us make the sign convention that the upward direction, vertically upward direction is positive. So in this case our force will be acting in the weight is in the, in the vertically downward direction. So let us put a minus sign in front of it. Then this, now we are going to assume that at t=0, the initial velocity was 0, which is the true case. Then we have v_f.

So in this case, because of the existence of this second term, the situation changes completely. So you can see that the smaller this interval, the higher is the final speed. So this is the reason when the rocket takes off from the ground, so the strategy is to burn a lot of fuel in a very short amount of time and so that the rocket can get a considerable speed. So, remember that to escape Earth's gravity field, the, we need to achieve an escape velocity which is 11.2 kilometer per second, which is quite high and, but most of, sometimes I mean not always the rocket has to go completely beyond the Earth's gravity field.

For example, these rockets are normally used to launch satellites in low Earth, what is called low Earth orbits, which are like few hundred kilometers from the Earth's ground. So the velocity of the v_f can be less than the escape velocity, but still it has to be considerable velocity, so that it can actually reach a very high altitude of few hundred kilometers at least. So let us take a numerical example to sort of get a feel about the number. So this is about a very big rocket.

Consider a very big rocket with total mass before take-off *M=3000 tons*, Mass of fuel in first stage *m = 2100 tons*, which gets burnt in *168 s*, exhaust speed of the fuel *u=2600 m/s.* Compute the thrust force.

So note two things about this. So note this word first stage. So usually when the rockets are designed, so it is designed, the fuel is kept into, ejected in stages.

So this is a trick to increase the final velocity v_f. So there are first stage, second stage, third stage and so on. Second thing is that, note that out of 3000 ton, 2100 ton is a fuel. So most of the rocket it tells you that most of the rocket is, mass of most of the rocket is just a fuel and this is required to just to lift the rocket by overcoming Earth's gravity. And this fuel gets burnt in 168 second, in less than 3 minutes. And the exhaust speed, the speed by at which the fuel is ejecting from the rocket is 2600 meter per second.

So how much this is in kilometer? So, 2500 meters per second is about 10000 kilometer per hour. So it is a fantastically large velocity. It is faster than speed of sound. Now compute the thrust force. So what is the thrust force? Thrust force is this quantity.

$$2600 \text{ m/s} \times \frac{18}{5} \approx 10{,}000 \text{ Km ph.}$$

$\hat{y}$

$$\vec{u} = -2600 \text{ m/s } \hat{y}$$

$$\frac{dM}{dt} = \frac{2100 \text{ Tn}}{168 \text{ s}} = \frac{2.1\times10^{6}}{168} \text{ Kg/s}$$

$$\vec{u}\frac{dM}{dt} = ?$$

So this term (**u**dM/dt) is called the thrust force. The thrust that is put on the rocket by the ejected fuel. So the problem basically asks you to calculate, so u is given 2600 meter per second, so in the negative y direction. So let us consider this as our positive direction. So the, this is in the negative direction. And it is given that the rate of change of the rocket's mass is given by, so 2100 ton by 168, it gets burnt in just 168 second.

So then the problem is basically ask you to calculate the thrust force. So I will leave it as an exercise, take home exercise for you to sort of insert the numbers and calculate the output. And also compare this number with the weight of the rocket and see whether this force is able to, is bigger than the weight of the rocket, only then it will be able to lift the rocket from the ground. So, in the next lecture, we are going to discuss more interesting examples of these conservation laws and energies and application of Newton's laws.

So we are going to discuss about the collision problems. See you next lecture. Thank you.