**Newtonian Mechanics With Examples**

**Shiladitya Sengupta**

**Department of Physics**

**IIT Roorkee**

**Week -01**

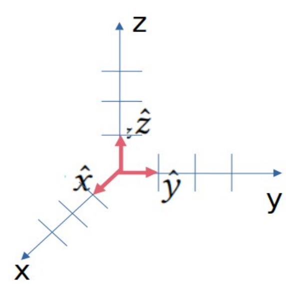
**Lecture - 03**

Welcome to this next set of lectures for the topic of scalars, vectors and tensors. So, in the previous couple of lectures what we saw is that we reviewed the definitions of what is scalars, and what is vectors, and we gave a most general set of definitions using which we can also include more complicated objects such as tensors. Today and the next lecture, the plan of this couple of lectures is, to now apply these concepts to do some calculations with vectors. Now, when I say calculations, what I am going to review are these elementary algebraic operations, namely the addition, and then we will talk about the products or multiplications between vectors. Now what about subtraction? So, we will see that subtracting two vectors is nothing but adding one vector to the opposite of the second vector. So, this is included in the operation of addition.

What about division? So, this is a remark that there is no meaning of the operation of division between two vectors because you cannot divide direction with another direction. So division is an illegal operation. For vectors, we will see that there are additions and multiplications or products, and in within the product, we will see that there are different kinds of products we can define for vectors, and we will go through them one by one. So, before we go to the actual calculations, we need the concept of unit vectors.

So, we are quickly going to review this concept. So, a unit vector is basically mean a vector with whose magnitude is 1. So, unit means 1. So, a unit vector is a vector whose magnitude is 1. So let us take an example.

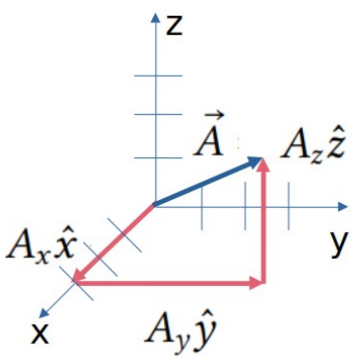
So, for example, if I take this coordinate system. So I have drawn a Cartesian coordinate system. So, this is the origin. This is the x, y, and z-axis. Now I want to specify the direction or orientation of this axis. So, I have drawn a vector, and this arrow represents this vector.



So, this vector the length of this vector is 1, 1 unit. So, if x represents length, then it could be 1 meter or 1 centimeter what is the appropriate unit for the problem. If the x represents force, then it could be 1 Newton, 1 micro-Newton and so on. So, this is 1 unit for the time being we assume. The direction of this vector represents the direction of the x-axis.

So, this symbol, note the symbol, so this symbol is a little hat, it is called hat or carat. So, this denotes the unit vector. So, in general, the convention is an arrow represents a vector, which can have any magnitude and this hat represents a unit vector whose magnitude is 1.

So now let us take a vector A, which is along any direction. So, this is our vector A. Then what is the unit vector along A? The unit vector along A is a vector whose magnitude is 1. its direction is the same as that of A. So, this represents the direction of A. So, as we seen in the previous set of lectures, we can write the A as a sum over some vector addition over three little pieces, one along the x-axis, one along the y-axis, and another along z-axis. And this Ax, Ay, Az represents the corresponding components along x, y and the z-direction.




$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\boxed{\hat{A} = \frac{\vec{A}}{|\vec{A}|} \qquad |\hat{A}| = 1}$$

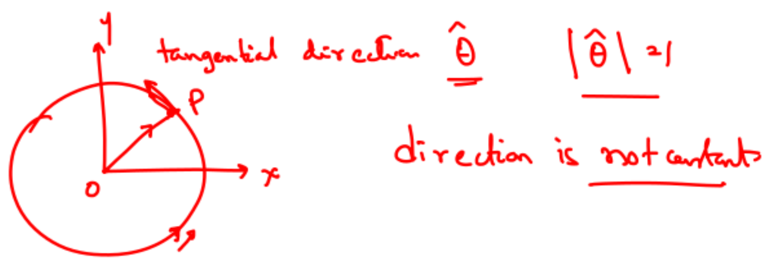
Then, the magnitude of the vector is given by this formula. So, if you know the component of the vectors Ax, Ay, Az then it is a simple application of Pythagoras' rule that the magnitude of the vector is the distance between the origin and the, distance between the tail of the vector and the head of the vector is given by Ax square plus Ay square plus Az square. So here the tail of the vector is situated at the origin. Then, if I divide the vector by its magnitude, remember that the magnitude is a real number. And I am assuming that it is not 0.

So, by real number I mean this is a scalar. So, remember the definition of scalar, so this magnitude of the vector does not change if I do a coordinate transformation or rotation as we took in the last class. So, if I divide by magnitude, then the vector that we got, which is called the unit vector will have a magnitude 1. Now I want to point out one important fact for which we will take a couple of examples in 2D.

So let us say that in 2D, we first draw, suppose we take a particle which is, moving along a straight line which means the direction is not changing, it is constant. So, then if we represent a unit vector, so suppose this is my x-axis then we represent the unit vector by x hat, so its magnitude is 1 and its direction is constant.


So, we can similarly draw another unit vector along the y-axis magnitude will be 1 and the direction will be constant. So, this is like a particle which is moving in a straight line.

But in general, the direction of the unit vector need not be constant. For example, consider a circle and consider a particle is moving along a circle or a person is walking along let us say the circular path.




So, in this case we can represent the direction in which the person is moving. So let us say we take the centre of the circle as origin. This is my x-axis, and this is my y-axis and let's say this, P is the location of the person at some point of time. Then this is the vector OP, which represents the position of the person at some point of time, P. Then this tangential direction is the direction along which the person is moving. It represents the direction of the velocity of the person.

So, in this tangential direction, we can denote a unit vector whose magnitude is one and which direction represents the tangent direction at any point of time where the person is located. So, this is usually denoted by the symbol θ and this θ hat represents the unit vector along the tangential direction. So, the magnitude of θ hat is one but the direction is not constant, it is continuously changing.

So, when we say unit vector it does not mean that anything about the direction, only the magnitude is 1. So now we are going to work through a series of examples to illustrate the different elementary vector operations, the additions, and the products. So, this is our first example.

If $\vec{A} = 10\hat{x} - 4\hat{y} + 6\hat{z}$ and $\vec{B} = 2\hat{x} + \hat{y}$, find

(a) component of $\vec{A}$ along $\hat{y}$

(b) magnitude of $3\vec{A} - \vec{B}$

(c) a unit vector along $\vec{A} + 2\vec{B}$

So let us work out one by one. So, the two vectors A and B are given and they are given in terms of the components. So just by the information that is given, 10 represents the component along x direction, -4 represents the component along y direction and 6 represents the component along the z direction. So accordingly, the answer is -4. So minus 4 is the component of A along y direction.

If $\vec{A} = 10\hat{x} - 4\hat{y} + 6\hat{z}$ and $\vec{B} = 2\hat{x} + \hat{y}$, find

(a) component of $\vec{A}$ along $\hat{y}$ $\rightarrow$ $-4$

Vector x, y and z these are the unit vectors along the coordinate axis. Next part,

If $\vec{A} = 10\hat{x} - 4\hat{y} + 6\hat{z}$ and $\vec{B} = 2\hat{x} + \hat{y}$, find

(b) magnitude of $3\vec{A} - \vec{B}$

So let us do it step by step. So, A is given, so first, what is 3A? So, 3 is just a number. It is a scalar. So, we multiply a vector with a scalar. It does not change the direction of the vector. It just changes the magnitude. Now minus B, so you can think of minus B as multiplying B with a minus 1. So, if this represents B, the length of the arrow is the magnitude of B, and the direction of the arrow represents the direction of B. Then, -B is an arrow in the opposite direction. So, the length remains the same, but the direction is opposite.

$$3(\bar{A}) = 3(10\hat{x} - 4\hat{y} + 6\hat{z})$$
$$= 30\hat{x} - 12\hat{y} + 18\hat{z}$$
$$-\bar{B} = -2\hat{x} - \hat{y}$$
$$3\bar{A} - \bar{B} = 28\hat{x} - 13\hat{y} + 18\hat{z}$$
$$|3\bar{A} - \bar{B}| = \sqrt{(28)^2 + (-13)^2 + 18^2}$$

$$= \sqrt{784 + 169 + 324}$$
$$= \sqrt{1277}$$

Okay, next question,

If $\vec{A} = 10\hat{x} - 4\hat{y} + 6\hat{z}$ and $\vec{B} = 2\hat{x} + \hat{y}$, find

(c) a unit vector along $\vec{A} + 2\vec{B}$

So, we write down the vector A, and then we write down the vector 2B. Then we will add this. Now if, I divide this sum by the magnitude of A +2B, that represents the unit vector along A+2B.

$$\bar{A} = 10\hat{x} - 4\hat{y} + 6\hat{z}$$
$$2\bar{B} = 4\hat{x} + 2\hat{y}$$
$$\overline{A+2B} = 14\hat{x} - 2\hat{y} + 6\hat{z}$$
$$|\bar{A} + 2\bar{B}| = \sqrt{14^2 + (-2)^2 + 6^2}$$
$$= \sqrt{196 + 4 + 36}$$
$$= \sqrt{236}$$

$$\frac{\vec{A} + 2\vec{B}}{|\vec{A} + 2\vec{B}|} = \frac{\cancel{10}\,14}{\sqrt{236}}\hat{x} - \frac{2}{\sqrt{236}}\hat{y} + \frac{6}{\sqrt{236}}\hat{z}$$

Note unit along A+2Bnis a vector, which has the same direction as A+2B and the magnitude, by definition, is always positive. So, we always take the positive square root. You can verify that if you apply the magnitude of this vector on the right-hand side is 1. The next example is-

Points $P$ and $Q$ are located at $(0, 2, 4)$ and $(-3, 1, 5)$. Calculate

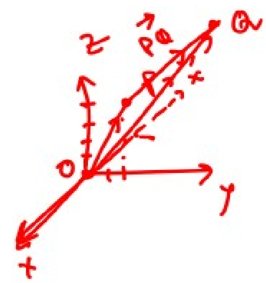
(a) The distance vector $\overrightarrow{PQ}$ from $P$ to $Q$

(b) the distance between $P$ and $Q$

(c) a vector parallel to $\overrightarrow{PQ}$ with magnitude 10.

Okay, let us do it one by one. So here is an important remark. So, when you do a physics problem and vector analysis problem, in particular, do not just apply the formula, do not start by thinking what is the right formula to apply. Start by drawing pictures that sort of give you a good understanding of what is there in the problem and it helps you to avoid mistakes in the calculation.

So let us try to draw the location of this point P. So, we have this, suppose this is my x-axis, this is my origin, this is x axis, this is y axis and z axis. So, it says that the x component is 0, y component of point P is 2. Let us say, I go two-step along y direction and then the z component is 4, that means I need to go four steps along z direction. So, this point P is located in the yz plane, so this is point P.




And if this is my origin, So OP represents the position vector or location vector of point P. Similarly, so -3, so that is on the other side, inside the screen and 1, so you have to go one step along the y direction and then five steps along the z direction. So, point Q will be located somewhere in here (image), so remember this is in 3D plane, so something like this.

So then, from P to Q, this vector represents a vector from the distance vector from P to Q. Now, if I look at this picture, I can see that this PQ is by applying the laws of vector addition is OQ-OP.

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$
$$= (-3\hat{x} + \hat{y} + 5\hat{z}) - (2\hat{y} + 4\hat{z})$$
$$= -3\hat{x} - \hat{y} + \hat{z}$$

So OQ is given by the position vector of Q. Now, if I take the difference, then I get distance vector from P to Q.

So now I need to know the distance between P and Q. So, the distance between P and Q is the magnitude of this distance vector PQ.

$$PQ = |\vec{PQ}| = |-3\hat{x} - 1\hat{y} + \hat{z}|$$
$$= \sqrt{(-3)^2 + (-1)^2 + 1^2}$$
$$= \underline{\sqrt{11}} \rightarrow \text{magnitude}$$

So, this is the distance, so this is the magnitude, the scalar distance between P and Q.

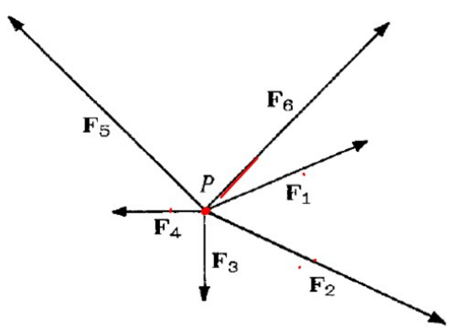
Now we want a vector which is parallel to PQ with magnitude 10. So now I want a vector A. So, A has a magnitude 10. So mod A is 10. So, this means that A is a vector, so the direction of the PQ can be represented by a unit vector along PQ. So, then the problem means that A is a vector whose magnitude is 10 and whose direction is along the direction of PQ represented by the unit vector along PQ. So let us calculate the unit vector along PQ.

$$\vec{A} \Rightarrow \quad |\bar{A}| = 10 \qquad \vec{PQ} = -3\hat{x} - \hat{y} + \hat{z}$$
$$\bar{A} \parallel \bar{PQ} \quad \Rightarrow \quad \underline{\widehat{PQ}} \qquad |\vec{PQ}| = \sqrt{11}$$
$$\widehat{PQ} = \frac{-3}{\sqrt{11}}\hat{x} - \frac{1}{\sqrt{11}}\hat{y} + \frac{1}{\sqrt{11}}\hat{z}$$
$$\bar{A} = 10\,\widehat{PQ}$$
$$= 10\left(\frac{-3}{\sqrt{11}}\hat{x} - \frac{\hat{y}}{\sqrt{11}} + \frac{\hat{z}}{\sqrt{11}}\right)$$

So, then the answer is the A is a vector, so this is the answer. You can again verify that the magnitude of this vector is 10.

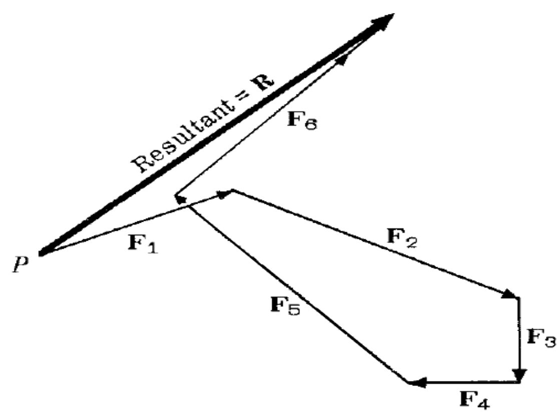
Our third example,

Forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$, $\mathbf{F}_6$ act on object $P$ as shown. What is the total (resultant) force acting on $P$?



So that means there are 6 forces. We have to find the sum of the 6 forces, F1, F2, F3, F4, F5, F6. This is the easiest way since the components are not given. So, the only way you can do this is by applying the vector law of addition to these pieces. Think of these each force as little piece of arrows, and then you start from anywhere.

So, take the first force for example, and then you take the second force, put it at the head of F1 and then you join the F3, the third force at the head of F2 and so on as shown in this figure. And then, finally, when you join these forces remember that you are not changing the direction of the forces, and you are not changing the magnitude, that is the length of this arrow. So, all you are allowed to do is to sort of slide these arrows parallel to itself. And then you get the tip of the last force, which is F6, and the head of the first force, which is F1. So now, if you join these two points, that will give you the resultant force or the total force, which is R.




So, note here that the notation, so here I represent the forces by a bold font instead of an arrow. So, the bold font means the force is a vector. So, in this case, the main lesson or main point to remember in this problem, the order in which we add the forces is immaterial.

So here I have added F1+F2+F3+ F4+F5+F6 but you can switch the order. You can add F2 to F3 and then F6, and then F5. So, in any order you can, so when you do a vector addition, the order in which you add the vectors does not matter. So next, we go to the multiplication of vectors or, sometimes called products of vectors. We will discuss three kinds of products in this lecture, set of this and the subsequent lectures. So, the first product is called the dot product.

Now what we are doing here is that we have two vectors. And if you have two vectors and by combining these two vectors, what is the resultant combination? So, we will see that there are two possible combinations. So many different kinds of combination is possible. So, the dot product is an operation or a rule which you apply on two vectors and the result is a scalar. So, you make a scalar out of two vectors. So, this is dot product.

So, suppose you have two vectors A and B. So let us say we write the A in the Cartesian components. And similarly, for B.

$$\vec{A} = A_x\,\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\,\hat{x} + B_y\hat{y} + B_z\hat{z}$$

Now, the symbol A without the arrow represents the magnitude of A and similarly for symbol B. So, with arrow represents the vector, without arrow represents the magnitude. θ is the angle between A and B.

$$A = |\vec{A}| \qquad \theta \text{ is angle between } \vec{A} \text{ and } \vec{B}$$
$$B = |\vec{B}|$$

Then A dot B, so this is the notation for a dot product in the middle. So, A dot B is defined as the magnitude of A times magnitude of B times the cosine of the angle between A and B.

$$\vec{A} \cdot \vec{B} = A\,B\,\cos\theta$$
$$= A_xB_x + A_yB_y + A_zB_z$$

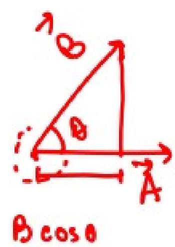
That is the rule. And we can write the same, same product directly in terms of components. So, it is x component times x component + y component times y component+ z component times z component

So, this is the rule. Now, in this course, some quantities that are dot products will be work done and power.

So, we will discuss those quantities in appropriate places. Here, we focus on the mathematics. Now, here is my second point. So, when you apply or when you come across any formula, mathematical formula, the we just do not simply try to remember the formula by force. First try to ask yourself, does it make sense? This is a very important question that helps you understanding physics. And what are the strategies? So, the first strategy is to draw sketches.

As you know, there is a saying that a picture is worth of 1000 words. so drawing pictures helps you understand the concept. So, what does the dot product mean in pictures?

Suppose this is my vector A. So, the length of this arrow represents the magnitude A, and the direction of the arrow is the magnitude. And suppose this is the vector B. And θ is the angle between the two vectors. Note that, there are two angles possible, and by convention, we usually take the smaller angle as the angle between two vectors. Now, you drop a perpendicular from B on A. Then you see that this piece represents B cosθ. This B cosθ is sort of the contribution of B in the direction of A.

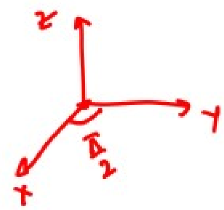



So, this is called the projection of B in the direction of A. So, then this A dot B is a number. First, it is a scalar. If you do a coordinate transformation, it remains invariant, so it is a scalar. It is a single number. It has the only magnitude, no direction. This represents the projection of B in the direction of A times A. So, this is kind of you are multiplying A, the magnitude of A by the projection of B along A.

But you can also do the opposite. So, for example, again the same picture. So, this is our B, this is the same angle θ. Now you drop a perpendicular from A to B and then you can see that this represents A cos θ. So, then the same thing you can write it as A cos θ times B, same dot product. So, then it will represent the projection of A along B times B.


So, this is the pictorial meaning of dot product. So, the next strategy, again, does it make sense? The next strategy is to consider special cases of the formula. So, in this formula A and B, sort of any general vector A and B, that does not give you any good picture. So let us take some special case. Let us take the unit vectors.

So, these are easy case because the magnitude is 1. So first, let us take an example. For simplicity, I am going to take the Cartesian components, Cartesian unit vectors. Suppose this is my origin, and I am choosing a Cartesian coordinate system to denote these vectors.




So, what is x hat dot x hat? So, if I apply this formula then the x hat magnitude is 1, x hat magnitude is again 1 and cosθ, the θ, the angle between the vector itself is 0. So, this is cos0 is 1. So, this gives you 1. And similarly for the other components.

$$\hat{x}\cdot\hat{x} = 1.1..1 = 1$$

$$\hat{x}\cdot\hat{y} = 1.1.\cos\frac{\pi}{2} = 0$$

Now if you take two different unit vectors x dot y hat for example, this angle θ is π/2 or 90 degrees. So, that means the x hat does not have any projection on y direction and vice-versa. That is y direction does not have any projection on x direction. So, that is why, they are perpendicular. If you take two perpendicular vectors, then their dot product is 0.

Now, we can apply this concept of product to calculate or to define the components of a vector. So, suppose I take a vector A. Now let us calculate a dot product between this vector times the unit vector along x direction.

$$\bar{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\bar{A}\cdot\hat{x} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot\hat{x}$$

$$= \underline{\underline{A_x}}$$

Now, from the previous calculation, So, x.x=1, So, you will get Ax and y.x=0 and z.x=0. So, you just get Ax. So, Ax represents the projection of A along x direction and, similarly Ay and Az. So, this gives you a pictorial meaning of what is the meaning of the components of a vector.

So, let us take an example.

Given vectors $\vec{A} = 3\hat{x} + 4\hat{y} + \hat{z}$ and $\vec{B} = 2\hat{y} - 5\hat{z}$, find the angle between $\vec{A}$ and $\vec{B}$.

How to find the angle?

So, if I apply this definition of the product, so this gives you AB cos θ and again, I can write the same thing in terms of the components, AxBx + AyBy + Az Bz. From the given problem Ax, Ay, etc, given, so I can calculate this quantity. From the components, I can calculate also the magnitude of the vectors and the magnitude of vector B.

So hence we can get from writing the dot product formula in two different ways and equating them we can compute the angle. So let us proceed.

$$\bar{A}\cdot\bar{B} = A.B.\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

$$A = \sqrt{3^2+4^2+1^2} = \sqrt{26}$$

$$B = \sqrt{0^2+2^2+(-5)^2} = \sqrt{29}$$

$$3.0 + 4.2 + 1.(-5) = 3$$

$$\sqrt{26}.\sqrt{29}.\cos\theta = 3 \qquad \cos\theta = \cos(2\pi-\theta)$$

$$\Rightarrow \cos\theta = \frac{3}{\sqrt{26\times 29}}$$

Now note one thing that, as you know, if you know the cosθ and then you are solving cosθ to calculate θ there are two angles, which has the same value of θ. So cosθ is equal to cos(2π-θ). So, in general you will get two solutions, one which is θ and one which is (2π-θ). So, by convention take the smaller angle and that is the angle between by convention the vectors A and B.

So, we will continue this lecture, this calculation. So, we will take more examples in the coming lectures. Thank you.