Course Name: Newtonian Mechanics With Examples
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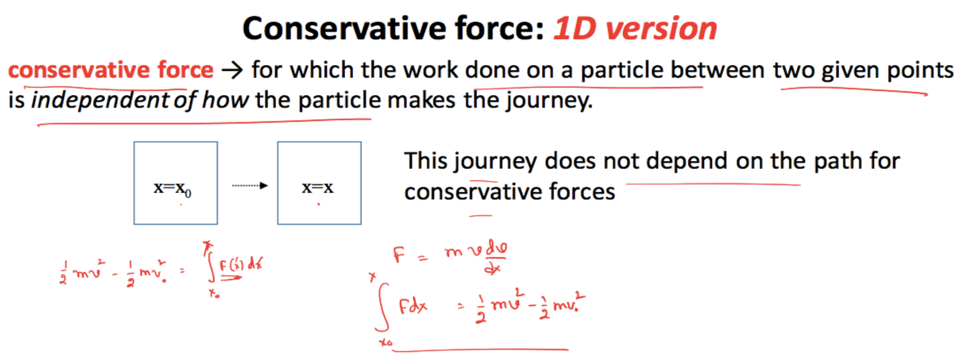
Week 06
Lecture - 28

Let us continue our discussion on the conservation of energy. So, in the last lecture, we took an example in which a car was moving horizontally in a highway. Let us now discuss in this today about some particle or something that moves in the perfectly vertical direction. So, what will happen? Let us say I have some pointer in my hand. What will happen if I can throw it perfectly vertically, upward? So, as you know from experience, let us say I throw it with some initial velocity v up in the upward direction. So, at this position, P equals 0, its kinetic energy is half m v square, and then we know that it will gradually reduce its speed, and after some time, it will reach a position of 0 speed, and then it will start to fall, and then it will come back to my original position.



Now, in this case, what are the forces acting on this object? So, of course, there is gravity, the attraction due to the earth, and then there is the drag force due to air. Now, imagine that we are going to, Suppose we ignore the drag on this particular object. So, let this is make reasonable sense if the speed of the object is very small. So, let us assume that there is only gravity.

It is the only force acting on this object. Then we know that as it goes up and stops and then falls, and when it comes back to its initial position, we are going to recover the same kinetic energy, which means that its speed at the initial point, When it returns to its initial position, it will be exactly the same as the speed with which it started. Only direction will be in the downward direction; the speed will be the same. Now that means that, so let us now ask that we learned from last class that the reason that the kinetic energy changes is because there is a force acting on it. Forced by the gravity acting on this particle, and this work done against those force, and that is why the kinetic energy changes.

So, this was the work energy theorem. So, the kinetic energy converts to the work done against gravity. And now, when it falls, this work is again gets converted into gravity because now the gravity is in the same direction as the speed. As a velocity. So, it is again converted back to kinetic energy, and when it comes back to the original position, If we ignore friction completely, we recover the original kinetic energy completely.

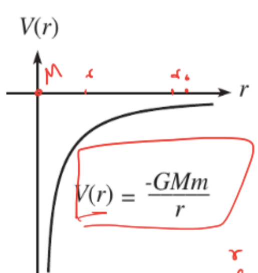
Now, this is a case in which we can simplify the work energy theorem even further. So, remember the work energy theorem, the way we wrote down in the last class was the half m v naught square minus half m v square is the work done. So, the change in kinetic energy is given by this particular extension. the work done on the particle. So, this is a two point equation.

Now, there is this special case of conservative force. So, these are the forces; there are certain forces for which we can simplify this even further. So, these forces, which are denoted as conservative forces, this has the property that the work done on a particle between two given point is independent of how the particle makes the journey. For example, it goes from this position x naught to x, this journey does not depend on the path of the conservative forces. Now if this is true, when can it be true? In 1D.

Potential energy function: *1D version*

Example (Gravitational potential energy)

Two point masses, *M* and *m*, separated by a distance *r*. What is the potential energy of the system at separation *r*?



W.E.T. $= \int_{r_0}^{r} F(r)\,dr = \int_{r_0}^{r} -\frac{GMm}{r^2}\,dr = \frac{GMm}{r} - \frac{GMm}{r_0}$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{GMm}{r} - \frac{GMm}{r_0}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_0^2 - \frac{GMm}{r_0}$$

$$\frac{1}{2}mv^2 + V(r) = E = \text{INVARIANT}$$

$V(r) = -\frac{GMm}{r}$ ⇒ potential energy function

$$\int_{r_0}^{r} F(r)\,dr = V(r) - V(r_0) \Rightarrow F(r) = -\frac{dV}{dr}$$

So, in 1D the options are limited because the path is a unique path in 1D; you can go from point 1 to point B, point 2, by a single unique path. Now, from our calculation before, we see that in if the force is only position independent, then for that force, this statement must always be true, that it must be a conservative force. So, this comes from our derivation of this theorem that if the way we derived that this F was a m v dv dx and then this work done was x naught times x, which is half m v square minus half m v naught square. So, if F is a function of x, then this integral will depend only on the end point; This is the essential mathematical point. Now, are there examples of such force? Well, Of course, I can give two common examples.

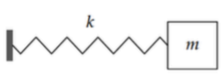
The first is the weight of an object, Mg. So, this is a force that is constant and remember that when I say force is a vector, so the force is a vector, so if force is constant, it must be constant in both magnitude and direction. Which is true for weight. Another common example is spring force. So, if we have this kind of force, which are called conservative force, we can define a function called the potential energy function.

And in this case, we can simplify, So this expression for work energy theorem, which is between two points and a single point, For every single point, we can define an invariant quantity, which is going to be the total energy of the system. So, let us try to understand this through an example. Let us try to work out an example. So, suppose there are two point masses M, capital M and small m, so imagine that the capital M is sitting at the origin and r represents the distance between the small m, the mass point small m, and the capital M, the distance separation between the two objects. So, if they are separated by a distance r, what is the potential energy of the system at separation r? So, what do you mean by that? So, let us start with this: we are going to calculate this work done for.

Let us see if we go from some point r naught to some point F, then What is the work done? So, the separation changes from some distance r, r naught to some distance r. Let us say this is r, this is r naught. So, how much is the work done by the forces? And this is the force; what is the force between these two objects? This is the gravity force and which is given by the Newton's gravitational law. So, we are going to calculate this and we know that if the separation is r, then the force between the two objects will be given by Newton's law of universal gravitation. And note that in this case, this is completely a function of only the position or In this case, the separation between the particles.

Potential energy function: *1D version*

Example: spring force



A point mass *m*, connected to a spring with spring constant *k*. What is the potential energy of the system at spring stretch *x*? (*x=0* denotes unstretched, equilibrium length of the spring)

$$F(x) = -kx$$

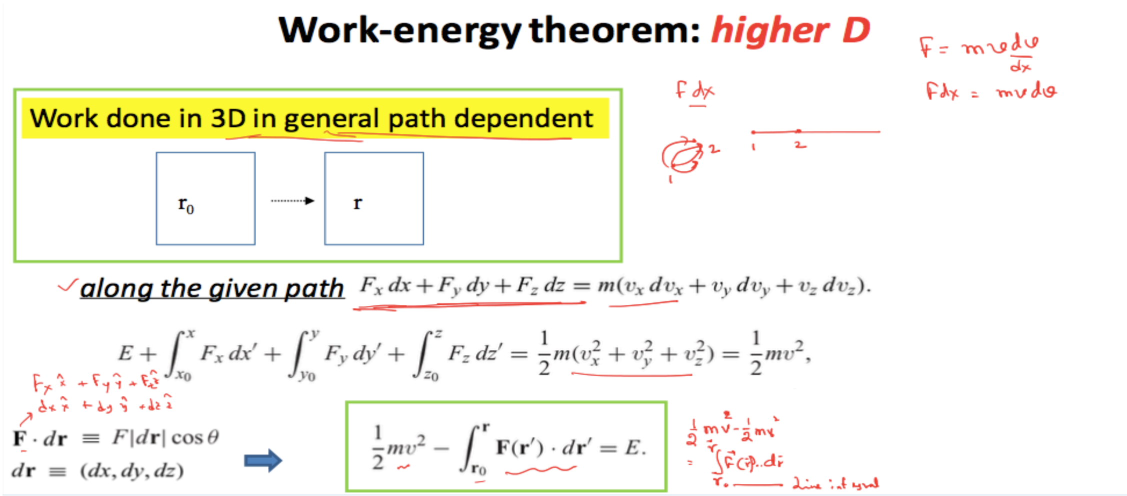
$$\int_{x_0}^{x} F(x)\,dx = \int_{x_0}^{x} -kx\,dx = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}kx_0^2 - \frac{1}{2}kx^2$$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \underbrace{\frac{1}{2}mv_0^2}_{KE} + \underbrace{\frac{1}{2}kx_0^2}_{PE} = E \quad \text{Total energy}$$

And if we calculate this integral, we get. So, now if we combine it with the work energy theorem, so between these two points, the kinetic energy also changes because of the work done by the force of gravity between them and by work energy theorem, so this must be equal to half m v square minus half. So, by work energy theorem, they must be equal and hence this must be equal to.

So, if we rearrange term, then we see that we can rearrange terms for every position of the particle. We can define a quantity which can be computed only by the information of that particle position. For example, at position r, this speed is v and the separation is r, At the separation r, so this left-hand side, it depends only on this separation r, does not contain anything about the r naught. And similarly, right-hand side does not contain anything about r, Only contain about separation r naught. And traditionally, this is written as a function where v r is given by.



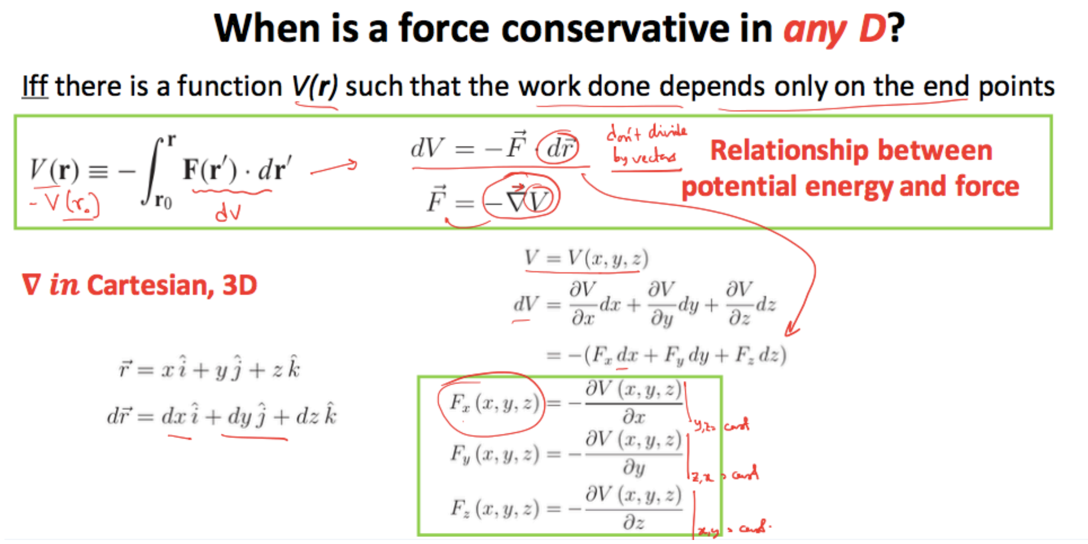
And this function Which can be defined for any position or in this case, the separation between the distance between the two objects as the potential energy function. So, this is a function Such that if we integrate, then we can, using this definition, write this work term. So, there is an integral such that the derivative, if we take the derivative of this function, We get the force or negative of the force, which you can verify. So, this is the potential energy function and using this concept of potential energy function, Which is the integral related to the integral of the work done by the force, then at each point, We can define a quantity. So, we see that the value of this quantity is same at point r as well as point r naught.

So, this is an invariant, which is called the total energy and This is an example of an invariant. So, this is the answer. So, this potential energy is given by this particular expression and using this particular expression, We calculate the invariant quantity, which is denoted as the total energy, the kinetic energy, plus potential energy as the total energy of the system. And remember that when you apply the law of conservation of energy, the choice of system can be very tricky. So, you have to define very, very carefully which forces are there and what is your system and what is the, what is the force for which you are calculating the potential energy.

So, let us try to clarify it further using another example. So, this time we are taking a spring. A point mass of small m is connected to a spring with spring constant k. What is the potential energy of the system at this stretch of the spring, where is x? So, x equal to 0 denotes the unstretched spring. So, when the spring is at equilibrium, If you stretch it, then it will try to come back to equilibrium and hence the mass m fills a force, which is given by, we know, which is proportional to the stretch x and it is, and the proportionality constant is called the spring constant and it denotes direct, the minus sign denotes that it is the direction opposite in which It is the opposite of x.

So, the sign is opposite to the sign of x. Now, if we apply the work energy theorem, let us say that we want to calculate the work done on the particle against this spring force and then this work done is given by, So, now if we equate this to the change in kinetic energy,. And then, if we rearrange term, we find that we can define a function for a single point. So, this is the kinetic energy. So, we are just rearranging the term from one side to another side.

So, the left-hand side is a function of only the point x and the right-hand side is a function of only the position where the stretch is x-naught. And this is the kinetic energy, this is the potential energy, and this is the total energy. Again, this total energy is an invariant, which is the same for all these different points. So, again you see that if we try to analyze the motion of the spring directly using Solving Newton's law, we have to follow the position and velocity of the mass frame by frame at every instant. But the conservation of energy gives us a shortcut, So that we can take any two position of the spring: equilibrium position, extreme length position, any position and connect them by an equation and this equation represents the physical fact that there is something called total energy, which is invariant and does not change as the spring changes its position.



the mass changes, the spring changes its stretching, or the mass changes position. Now, our next part of today's discussion is going to be a little bit mathematical. So you are trying to generalize the work concept of work energy theorem and the concept of conservation energy to higher than one dimension. So, the math's is going to be little bit more complicated. By higher, I mean 2D, 3D, and so on.

So, the issue is that suppose the same question We are asking that we have this system, and it has two positions: the first position is r naught as some initial position and the final position is r and when to ask how much and there is a force acting on the particle or our object system and the question is: What is the work done on the particle? Now, in 3D, for example or any or even 2D, the difference is now that earlier our expression of work was just F times displacement, but Now there are many different ways. So, this is the key point: when you go to 2D or 3D There are many different ways to go from point 1 to point 2. So, the path is now no longer unique. So, in 1D, there is only one way to go from point 1 to point 2 If you are moving along a line, but in 2D, for example, on this plane, there are infinite number of ways, In 3D, there are also infinite number of ways and more ways in 3D than in 2D to go from point 1 to a point 2. In that case what we have to keep in mind is when you calculate the work done, We need to know which path the particle followed.

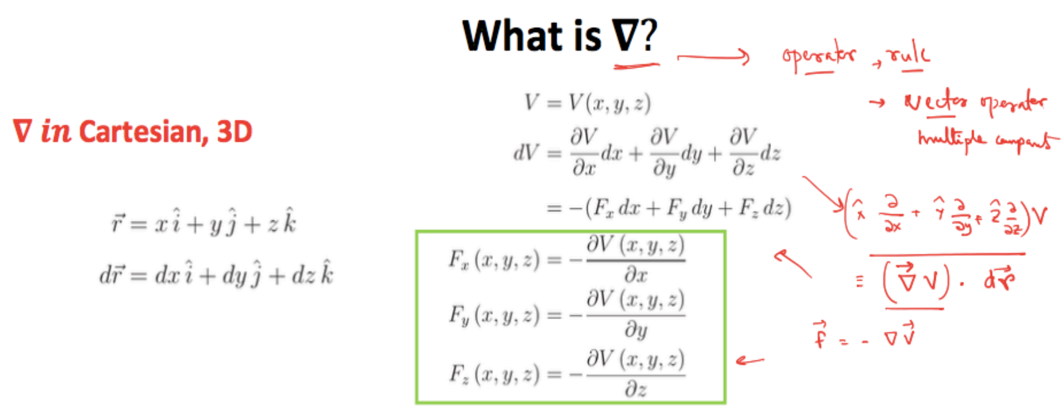
So, the work done in 3D, or in any higher D, even in 2D, is in general path-dependent. So, if we specify the path, then the total work, so we can generalize the expression for work. So, instead of one single piece of displacement along, let us say, the x direction, Now we have three independent pieces: one corresponds to the x direction, and the second movement is along y direction and third movement along the z direction. Why three independent spaces? Because any displacement in 3D, So, here we are going to take the expression of an example of 3D. We can always compose the displacement into three components: one along x direction, one along y direction, and one along the z direction, where x, y, and z are mutually perpendicular axis.

So then our expression for work will consist of three pieces; the first piece is the moving along the x, E 3 one dimensional piece, and you have to sum over them. And from this is one side of the equation and we got this expression that F is M v x, so this is dv dx in 1D, so F times d x is equal to M v d v. So, this is true now for each of the separate x along this x, y, z separate direction separately, so on the right hand side will be similarly composed of three; sum over three pieces, one for the x direction, one for the y direction, and one for the z direction. And hence going from a point R naught to R is equivalent to the x naught coordinate x naught changing from x naught to x, y changing from y naught to y, and z changing from z naught to z. And then we can sort of have, So, if we do the integral, then we get an expression for kinetic energy again.

Now, this algebra becomes more elegant if we use our vector notation, so if we use our in vector notation, So this work done, so the expression that we get for work, Looking at it, it is obvious that it can be expressed as a dot product between the two vectors. Vector F with component F x, F y and F z and a vector d r, which represents the displacement with components d x, d y, and d z. So, if we take dot product of these two vectors, You can easily verify that we get back this particular expression. So, then, in this case, the work done between two points, So, we have this work done. This is the work done on the particle; this is the kinetic energy of the particle.

So, between two points, we get this particular expression that half m v square minus half m v naught square is equal to the part done on the particle, which is now a integral, A strange integral in 3D, so this is called a path integral, so a line integral, So, we have to specify a particular root by which the particle goes from R naught to R and then you can compute this quantity and because in 3D or even in 2D, there is no unique path, So, this right-hand side is, in general, depend on the root that the particle takes and hence the kinetic energy change will also depend on the root that the particle changes. So, this is the generalization of work energy theorem in 3D, Which works also in 2D, so again, the point is that I emphasize The key issue is that in higher dimension, higher than one dimension, The path is not unique, but we can still ask this question that in like in 1D we took two examples like the gravitational force and the spring force for which this integral of work done were Independent of the path followed, it depends only on the end points, So, we could define the potential energy function so that the integral, this particular integral, The value of this integral depended only on the end points x naught and x. Now, is it true in 3D or 2D also? So, the answer is yes, So, we can also have conservative force for which this property is true: that this journey going from point A to point B does not depend on the path that we follow, so it will depend only on the end point of this path, so there are certain special forces for which indeed it is true even in higher dimensions. So, the question is that in higher dimension or any dimension When is a force conservative? So, you are going to show this mathematics and this analysis using the, so you are going to sort of use it as kind of an exercise in vector analysis. So, the condition is that if there exist a function v r, which is a potential, Which is anticipated to be the potential energy function such that the work done depends only on the end points, then we will have a conservative force.

So, this is the definition of the potential energy function in 3D. So, now the work done for a displaced, small displacement d r is given by this dot product between the force times displacement both of which are vector and the overall work done along a path from r naught to r is going to be the difference in the potential energy. So, there exists a function such that this work done will depend only on the end points and the values of the end points of the potential energy function at the end points. Then, if we compare, we can say that this small change this can only happen if this small work done, infinitesimal work done at certain points r prime between r naught and r must represent in small change in the potential energy. So, Now since this is a vector, you cannot divide a scalar d v, which is the change in some energy a scalar is a single number You cannot divide both sides by a vector.

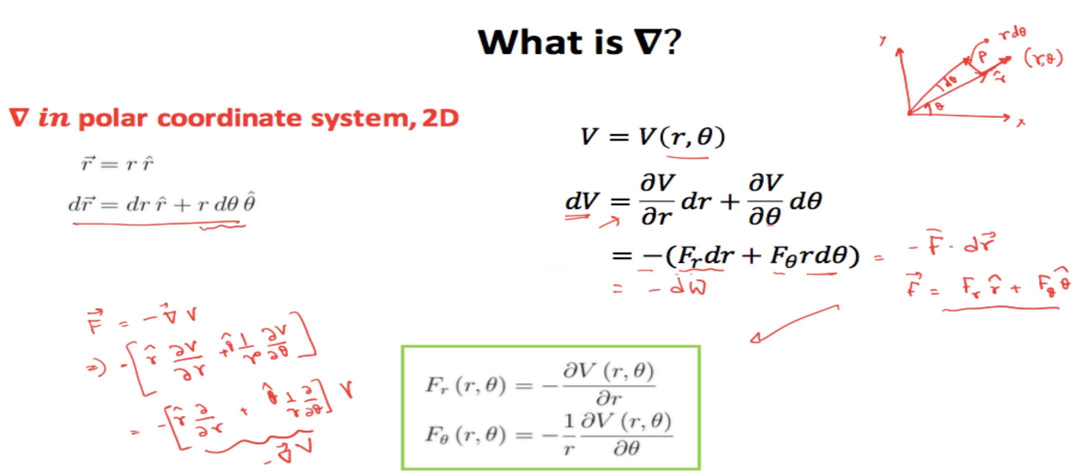


This is very important. Instead, we define a function, which is defined new symbol, which is inverted triangle with a vector, is called a del operator or a vector operator. So, think of it as some kind of rule with multiple components like a vector that acts on a scalar and gives you a vector. So, what is this operator del? So this is called del operator. So, let us understand this. So, I am going to take first the Cartesian components and in 3D.

So, suppose in Cartesian components it is function v, Which you can define at each point in space. So, each point in space is denoted by three numbers three coordinates of that position: x, y z. So, v is a function of the position of points in space. Then, by calculus elementary definition of calculus, a small change in the function TV is given by the right-hand side expression and if we compare these two expressions, we arrive at the relation between the components of the force (x, y, and z) and the derivative of this potential. So, the x component of force is nothing but the negative of the partial derivative of v with respect to x, and so on for the other components.

So, note that this is now a partial derivative, which means taking this derivative and keeping y z equal to constant. This means that the taking the derivative with respect to y and taking the derivative keeping z and x equal to constants. This is the meaning of partial derivative. Since we have now many different multi component function, multi variable function, So we cannot use a total derivative; we have to use a partial derivative. And in this particular case, the displacement is given by this vector, this is the displacement vector.

So, let us now take another example of an expression for del. So, okay, so we have already discussed the expression for del in this operator, so this del is called an operator. Which means it is a rule and has multiple components. So, it has a components which is a vector. Note that a vector operator does not have a direction; it just means It is a rule with multiple components.



So, this one we say that imagine that if we take this, if we take this as if this is a rule, so the rule is to take derivatives. And this particular rule acts The function v is equal to the grad v and this is a rule, so the resultant quantity is a vector, and this can, now taken a dot product with displacement and that will produce this particular expression. So that if we compare it the work done, then the force is given by grad v where the components are worked out in 3D, in the Cartesian here. So now, just to show it further, so we also do not need to take Cartesian component.

We can also take other coordinate systems. For example, in let us for simplicity, Let us take a 2D coordinate system. So, in 2D, we can take the Cartesian coordinate system. but we can also take what is called a polar coordinate system. So, this is the origin and you join some point P with the origin, then the position vector of this point. So, the distance is r and the angle it makes with the positive of x axis is the orientation of this vector.

Now, with this and there, you can imagine that the position is now given by two numbers, r and theta. So, this potential energy function which is again defined at each position of a point particle, so it is a function of r and theta. Again, the calculus shows that elementary, small, infinitesimal change in v is given by this particular expression of the right-hand side. So, if we look at the displacement of a particle, now the displacement is not simply dr plus d theta, but dr which is a displacement component in the radial direction keeping theta constant and this is a displacement in the tangential direction. So, if this is a delta theta, it is exaggerated, and then this displacement is r times d theta.

It is not only d theta, but r theta multiplied by r. So, then the displacement in if you keep r constant, that this radius r constant and go in the tangent direction, then the displacement is r d theta times theta. So, If we now write down the expression for the work, then as F dot dr, then theta d theta, then we get a, so the F is that we expect that it must have a component in the r direction and radial direction, and a component in the tangential direction. So, this will be the two components of F and then this F times displacement and F r radial component times the displacement in the radial direction; that is the dot product. And then, by comparison, comparing this line with this line, we get an expression for the del operator. So, If you say that this force is minus gradient of v, then we can express it as r So, this will be the expression of this operator, del, which acts on the potential energy function to produce the force.

So, now coming back to our question that when, what is the condition that the force will be conservative in higher dimension, in higher than one dimension, or, In fact, this is also what we have discussed; it is going to be true for any dimension. So, way, we are going to state without proof, because the proof is beyond the scope of this particular course. But I am certain you are going to learn about this in a mathematics course on vector analysis or a electricity magnetism course. So, given a force F r, so F is a vector, so it is denoted as a bolt; tie it and r is also position; it is also a vector. So, necessary and sufficient condition for the potential function v r to be path-independent and well defined, so that that means that whether we can find a potential energy function, v, such that we can express this F integral of work done as a kind of proper, as just by given by the end points.

When is a force conservative in *any D*?

Given a force ***F(r)***, is necessary and sufficient condition for the potential function ***V(r)*** *to be path independent and well defined is* that *curl of* ***F*** is zero everywhere.

$$\text{Curl of } \mathbf{F} \equiv \nabla \times \mathbf{F}$$

$$\nabla \times \vec{F} \equiv \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$\nabla \times$ *in* Cartesian, 3D

So, there is a well-defined v r function that exists, which is a condition, The necessary and sufficient condition is that another operation is the curl of F, so this is the symbol of curl of F. So, again, the same operator del, and now this is a cross-product. So, if you act, there is a different rule, a different operator, different vector operator which acts on F for vector and produces another vector, which is called curl of F. And the condition is that for the existence of a potential energy function is that this curl of F must be 0 everywhere. So here, I just mentioned for the sake of completeness, how to write down this rule, the curl of F in a Cartesian components in 3D, which is just given here, so you can sort of work it out, but I am not going to say more about that.

Instead, let us try to illustrate it through an example. So, there is a very important class of forces called the central force. They are very important in three dimensions. So, their property is that this is a force such that, so suppose this is an origin and this is a point P. this is origin and the position of P with respect to this origin is R. So, then let us say something on P sits, let us say there is a point mass, something like a sun, which sits at O and Let P be the position of earth, Then there is a force acting on the particle or the point mass at P is of experience feeling a force and this force is such that the magnitude of the force, which is F without the vector sign, depends only on this distance R and it has no tangential component.

Example 23: Central force in 3D

A central force is defined to be force that points radially and whose magnitude depends only on r. That is, $\vec{F}(\vec{r}) = F(r)\,\hat{r}$

Show that $\nabla \times \vec{F} = 0$

r is a function of *x,y,z*. *F(r)* is a function of *r*. If *x, y* or *z* changes *F(r)* also changes. Think carefully w.r.t which variable you are taking derivatives!

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r}$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(yF/r)}{\partial x} - \frac{\partial(xF/r)}{\partial y}$$

$$= \left(\frac{y}{r}F'\frac{\partial r}{\partial x} - yF\frac{1}{r^2}\frac{\partial r}{\partial x}\right) - \left(\frac{x}{r}F'\frac{\partial r}{\partial y} - xF\frac{1}{r^2}\frac{\partial r}{\partial y}\right)$$

$$= \left(\frac{yxF'}{r^2} - \frac{yxF}{r^3}\right) - \left(\frac{xyF'}{r^2} - \frac{xyF}{r^3}\right) = 0.$$

So, the only component is radially. So, the F can be directed along O P either towards O P or away from O P. In the case of gravitational force, this attraction is attractive force, so it is directed towards sun, which is attracting earth towards itself. If we have two point charges, one at O and one at P, with the same, Let us say two electrons have the same charge; then this could be repulsive. The force on the electron at P will be pi, and the electron at O will be repulsive, so it is along R.

So, This force is along R or along minus R. Then, if this is the case, then this force is a conservative example of a conservative force, which means that there is a well-defined function, potential energy function, which we know exists because we know the gravitational force has a gravitational potential energy function and the Coulomb's force has an electrostatic potential function. So here, we just verify that if we have a force like this. Then the curl of F will be 0. Now, we are going to use this expression here, This rule is provided here to compute the curl of F, and it is sufficient to calculate only the x component. If you, I mean likewise, can show the other components will be similarly 0.

So, only thing to note is that you have to calculate this quantity x; this is the one component. This is the z component of the curl of F. So, the F y, so the, in the Cartesian coordinate system, this is F r and this r hat is given by: it is a unit vector. This is a unit vector, so this is the actual position vector, divided by the magnitude and this r, the position vector r, as you know in Cartesian, is given by this particular expression. So, r hat is given by, and then if I, so this is r hat x component, so F times x by r x hat plus y by r y hat plus z by r z hat, so F x is F x by r, F y is F y by r, and F z is F z by r.

So, now, we have to calculate this. Now plug in all these quantities, and then you just calculate the derivatives. There is only one thing you have to remember, so there is this identity that you should use. r is the square root of x square plus y square plus z square. So, suppose if we have del r del x.

Which is given by, so if I apply chain rule, we get x square plus y square plus z square to the power half, so half into 2x, because del y del x is 0, del z del x is 0.

So these two get cancelled; if it is 2, then this is r, so this is x by r. Which is written here, and similarly for the other components. Now, if you apply this rule and just plug in this chain rule, and then you will see that everything will get cancelled and You will end up at 0. So, for a central force, the curl of F is 0 everywhere. This force is, therefore, a conservative force.

So, I am, just to summarize what we discussed today. So, we discussed the conservation of energy, we discussed, generalized the work energy theorem from 1D to 2D and 3D and higher dimension, and we discussed conditions under which in higher, so the main issue is that the main difficulty when you go from 1D to higher D is that the path that a particle can take to go from point 1 to point 2 is no longer unique. Under this situation, we can still define a conservative force, and we discussed. What are the conditions for a conservative force to exist? We say that if the force is conservative. If the force is conservative, then what is the relation between the potential energy function and the force? And what is the condition that, and finally, we discussed that If the curl of the force is 0 everywhere, then the force must be conservative. In the next class, we are going to discuss conservation of momentum and the momentum balance principle. See you next time! Thank you very much.