Newtonian Mechanics With Examples

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Week -05
Lecture-26

So, we have been discussing the friction. In the last lecture, we learned about the form of drag force in different regimes of Reynolds number. In this lecture, we are going to apply those concepts to solve problems. So, consider a ball is dropped from rest at a height h, so in a viscous liquid. Assume that, this liquid could be some something like air or a fluid which can be air or some viscous liquid like glycerin or honey, which are typically used in what is called a Stokes law experiment for measuring the viscosity of liquid in a bit typical B.Tech first year lab physics experiment, the undergraduate physics experiment.

A ball is dropped from rest at height h. Assume that the drag force due to fluid takes the form of $F_d = -\alpha v$. Find the velocity and height as a function of time.

Example of dropped ball (in liquid) →
Stokes Law experiment in UG 1 Physics Lab – drop a ball inside glycerine

So, in last lecture, we find out, now we know under what condition we expect the drag force to take this particular form. So, the question is find the velocity and height as a function of time. So, here is the solution of how to do it. So, first let us say we take our coordinate system and let us say we start at some origin. And let us say the vertical direction is y axis and this is a one-dimensional problem.

So, we write down the Newton's law for this problem. So, what are the forces acting on the ball? So, let us say this is mass m, so then the force that is acting is the, its weight mg downwards and then we have this drag force. Now the sign, so we are going to use minus here, but the sign can be positive or negative depending on the sign of the velocity. So, v represents the velocity and then this must be equal to, so this is the total force and now this is an example where the system is no longer at rest, it is moving, it has some acceleration, and the initial condition is given at t=0, the coordinate of the position of the ball was y at h and its drop from rest.

Ex: dropping a ball

$$m\ddot{y} = -mg - m\alpha v$$

$$t=0: \quad y(0) = h$$

$$v(0) = \dot{y}(0) = 0 \quad \Big] \text{ initial condition}$$

Now, this equation is easy to solve and in fact this is a differential equation that all of you know how to solve. So, for example, you can write this

$$\ddot{y} = \frac{dv}{dt} = -g - \alpha v$$

$$\int_0^v \frac{dv'}{g + \alpha v'} = -\int_0^t dt'$$

$$\ln\left(1 + \frac{\alpha v}{g}\right) = -\alpha t$$

$$1 + \frac{\alpha v}{g} = e^{-\alpha t}$$

$$\dot{y}(t) = v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right)$$

$$-\frac{g}{\alpha} + \frac{g}{\alpha} e^{-\alpha t}$$

So you have a separation of variable and then you can integrate and the left-hand side, so let us say you are integrating from, so, you can separate the velocity and time and when integrating from time from t=0 to some time t and on the right hand side at t=0, the velocity was 0 at some time t later the velocity is v. So, if you integrate, you get a log function and then if you rearrange the term, you can solve for the velocity as a function of time and it turns out to have this particular expression and it is in the negative direction which means that the ball is going down because in our way we have taken the coordinate system, the upward direction is positive,

Once you know the velocity as a function of time, so this velocity is dy/dt, so now again you can integrate once more to compute the position as a function of time and it is also easy and standard integration, so just an integration of exponential and if you now plug in the condition that your t=0, y=h, then the final answer becomes this.

$$\int_h^{y(t)} dy' = -\frac{g}{\alpha}\int_0^t \left(1 - e^{-\alpha t'}\right) dt'$$

$$y(t) - h = -\frac{g}{\alpha}\left[t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right]$$

$$y(t) = h - \frac{g}{\alpha}t + \frac{g}{\alpha^2}\left(1 - e^{-\alpha t}\right)$$

Now this one is somewhat different from, so this answer is different as compared to what you normally expect from just a motion if you drop a ball and it falls with a constant acceleration, if the answer is somewhat different, it now has this new term. So, when you do this calculation, after you get some answer and suppose the answer is not given, then you have to ask yourself this question always, does it make sense, does the answer make sense? So, of course you can go and check the answer but if the answer is not there, so then what to do? So, here is a simple check point that works with this problem that suppose you ignore the drag force, so then you know what is the ideal result that you have all of you done in high school.

So, ignore the drag force. What does it mean? When can you ignore the drag force? You can look at this equation of motion. So, there are two forces, one force is a constant and the other force is proportional to velocity. So, ignore the drag force means ignore the drag force compared to the weight of the ball. Now this is possible when the velocity is small, when is the velocity small? At the initial time or close to the initial time because the ball is starting from rest.

When the time was small, then velocity was small and during that period perhaps you can ignore the drag force. Now how much time is small? So, note that the answer has this combination αt and this is sitting as a power of exponential which means this αt must be dimensionless, this you can check from here from the dimension analysis here.

$$[\alpha m v] = [\alpha]\cdot M \cdot \frac{L}{T} = \frac{M L}{T^2}$$

$$\Rightarrow [\alpha] = \frac{1}{T}$$

So, mαv has a dimension, must have a dimension of force and hence we see that α must have a dimension of inverse t that means αt is dimensionless. Then you can expand this small time means that times p small such that this product α t is very-very smaller than 1, so that is a measure of small time.

drag force negligible : ~~ie $\alpha v \ll g$.~~

small t: $\alpha t \ll 1$

$$\alpha t \ll 1$$

$$y(t) \simeq h - \frac{g}{\alpha}t + \frac{g}{\alpha^2}\left(1 - \left(1 - \alpha t + \frac{\alpha^2 t^2}{2}\right)\right)$$

$$\simeq h - \frac{gt^2}{2}$$

So, I chop this series at the higher and higher term has higher and higher order of αt and if αt is a number which is smaller than 1 then those numbers are smaller and smaller, so, I chop this series at the second term. And then if I simplify I get this answer, which is your standard answer that you get in absence of any drag force, then you have confidence in your answer. So, in presence of drag force your position is no longer given by this expression there is some additional term.

The more important part the effect of drag force is that what happens in the limit of large time. So, again large means we can have this dimensionless parameter αt, so, you can say

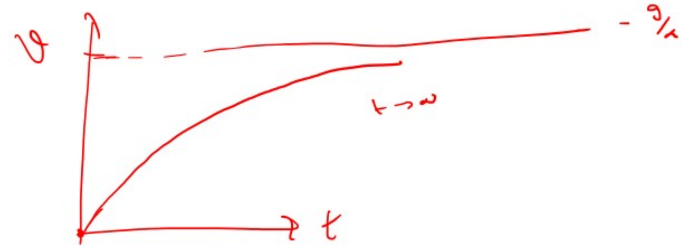
large t: $\alpha t \gg 1 \qquad e^{-\alpha t} \to 0$

$$\boxed{v(t) \to \underbrace{-\frac{g}{\alpha}}\ = \text{constant}}$$

for this value of v, $\ddot{y} = 0$
total force vanishes.

So what does it mean that the velocity is constant? Note that this is not exactly constant because αt is exactly 0 when t actually approaches infinity. So, we say that this goes to constant asymptotically which means if I plot the time the velocity the magnitude of velocity as a function of time so it starts from 0 and then gradually increase but its acceleration goes down and hence it approaches this limiting case as time tending to infinity and this value is -g/α.

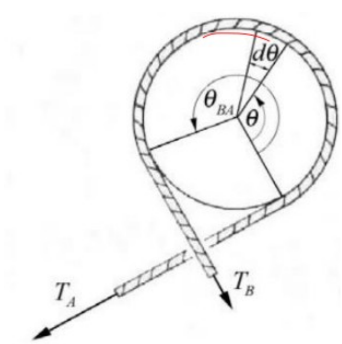
Now this velocity has a name it is called the ‘terminal velocity’. It basically means that what happens is that if you look at the Newton's law of motion So, there is a constant force and the force which is increasing with velocity but the rate of increase is slowing down. So as the difference between these two force and because v is stand also be negative so we know that this is a friction which is opposing the relative motion of the ball through the fluid which means this force is in the opposite direction opposite to the



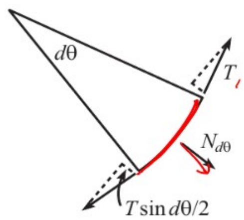
gravity. So then the overall force on the object is much less than the weight of the object itself and as the velocity increases it gradually approaches a scenario where the two forces are almost equal or equal to each other so they cancel each other. So, then if the total force is 0 that means we expect the ball to move with a constant velocity and this velocity has called the terminal velocity which has this particular expression and this is the reason why we survive after getting hit by a raindrop which is falling from a height of 1 kilometer. So this is a case where you see that the result of the motion is completely different in presence of drag force and in absence of drag force.

And so in so this is a scenario where in real life situation where you absolutely cannot ignore drag force when analyzing the motion of a projectile such as a raindrop. Now here I make one more comment. Okay before going to the next problem so this is the following. So, remember that in earlier we defined different types of mechanics problem and this particular example belongs to the second situation where we have the total force on the particle on the our system is not 0 and all the forces are given the initial condition initial location position and velocity of the particles are given so the equation of motion is given. The initial condition is given and our goal is to determine the trajectory of the particle. So, this problem fits into that particular situation so this introduces this very important concept of terminal velocity. Now we go through very quickly through another example which is sort of another example of friction and in this case this is a solid we are going back to the dry friction or the friction between two solids and some of you may have already familiar with this problem but I find it a very nice problem which illustrates the practical application of friction.

A rope wraps an angle θ around a pole. Pull at one end with a tension T_A. Other end is attached to a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force rope can exert on the boat, without the rope slipping?



So remember that when we review the nature of the tension force in a rope in a few weeks back there we sort of gave an example that the tension in a rope which we normally assume to be uniform need not be the case always. So, this is a situation where the tension in the rope along the rope which is wrapped around this pole is not uniform because of an external force on the acting on the rope which is the friction between the pole and the rope. So there is a coefficient of static friction between the rope and the pole is μ then what is the largest force rope can exert on the boat without the rope slipping so this is the problem.



$$N(d\theta) = 2T \sin \frac{d\theta}{2}$$
$$\simeq T\, d\theta$$

$$T(\theta + d\theta) \leq T(\theta) + dF_f$$
$$\leq T(\theta) + \mu T\, d\theta$$

$$dT \leq \mu T\, d\theta$$
$$\int d \ln T \leq \int \mu\, d\theta$$
$$\ln T/T_0 \leq \mu\theta \qquad T \leq T_0 e^{\mu\theta} = T$$
$$\boxed{T(\theta) \leq T(0)\, e^{\mu\theta}}$$

* why the ropes are so thick that ties the boat
* why we also put knot?
* what if the pole breaks?

Now I sort of show you the analysis here and this analysis is pretty standard so I am not going to sort of go through it quickly and it is worked out here so, but the point is that if you look at the rope as your system and then there are and one part of the rope slice of rope which makes an angle dθ let us say as your system. Then there are three forces so the tension force from the surrounding part of the rope plus a contact force the contact force of the rope and the pole which has two component one is the vertical component and other is the tangential component and because of this external contact force the frictional force.

T is non-uniform so it depends on the angle θ . The important point is that this tension depends on angle θ which is kind of intuitively obvious the more you the more length of the rope is contact with the force and more frictional force so you can intuitively expect that it is little bit more hard to pull at one end so the friction will try to prevent the rope from slipping and that is the whole point why we tie the rope around this pole to increase the length of the contact of the rope with the pole but what is important is that this happens in an exponential way so this is the important part of result from this calculation that if we look at one slice and do the force balance what we get is that the difference of the force so this difference in the tension on the two sides which is the frictional turns out to be exponential in θ so this is the crucial part and which is why this rope tying this wrapping this rope around the pole is very effective.

Now I invite you to think a couple of question which I am not going to discuss. But it will be fun to think from your part is that why the ropes the ropes that you tie for example if I want to tie something some small objects around.

If this is my pole, then I have a small rope. But the ropes that are used to tie boats or launch steamers or ships around in the harbor in the port. They are very thick. Why are they so thick? what does the thickness does to the rope? Second thing is that we not only wrap the rope around the pole we also tie some knot. What does the knot play a role what is the role of a knot and third question is what if the pole breaks.

So, I invite you to think about the interesting questions and in the next week we are going to discuss the new topic but before that let us summarize what we have discussed so far. So, we have looked at two kind friction so we have reviewed in this week We have reviewed the the two kinds of friction and basic coulombs law of friction between two solid surface and you also consider a case of friction a drag force between a solid and liquid and found its expression in different its different regimes of Reynolds number and we took some example and most importantly we found that this that when we consider the projectile motion the drag force is some often very significant and finally through different examples I try to convey the message that in real life there are many practical applications of friction they are very common and they are very useful sometimes useful sometimes not but they are very interesting to study. So thank you. See you next week.