Newtonian Mechanics With Examples

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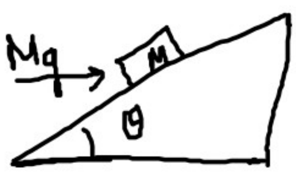
Lecture-24

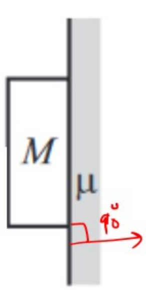
In the last lecture, we critically review the Coulomb's law of friction. Today, the plan is, in this lecture, the plan is to take some examples to illustrate the correct way of analyzing problem with friction. So, first example is something that is quite familiar to you. It is a block on an inclined plane problem, but I am sure that at the end of the day, end of this analysis, you will find something surprising about the same problem. So, the problem is the following.

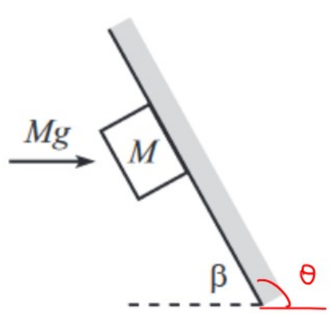


A block of mass M rests on a fixed plane inclined at an angle θ. You apply a horizontal force Mg on the block. Assume that the friction between the block and the plane is large enough to keep it at rest.

Q: If the coefficient of static friction is μ, for what range of angles θ will the block remain at rest?

So, this is the question. Now, before we start to analyze it, I want to mention one point that normally and as we have done also that when we draw this, think of this inclined plane, we assume that the value of the inclined plane, the angle of the inclined inclination is less than 90^0. But in real life, you often encounter inclined planes where the value of the inclined plane θ is actually 90^0 or can be even bigger than 90^0. For example, as shown in this figure, this angle is 90^0 and in this case, this is θ and this angle θ is actually greater than 90^0. Now, imagine that there is an ant which is sitting standing still on a wall.





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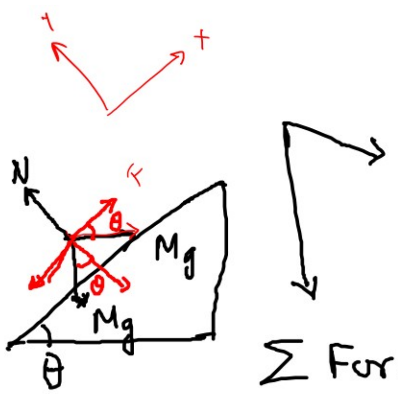
So, this vertical position could be something that is an ant on a wall. And similarly, this situation where angle is greater than 90^0. If you still are not still convinced that angle can be actually greater than 90^0. So, look at this picture of a tree branch and this is a kind of a natural example of an inclined plane. So, this tree branch represents the inclined plane and in this case, the angle is continuously varying and in fact, if you look at this part of the branch, the angle with the horizontal is actually greater than 90^0.

So, anything like a bird which is sitting on a tree branch at different position can be our block on an inclined plane problem. So, how do we analyze? So, let us analyze this situation using the free-body diagram method. So, we are going to take the block as our system. So, there are three forces from the surroundings. So, one is the earth.

System = Block

Surrounding:
- earth
- contact force
  - N
  - F
- ext. applied force Mg

So, this earth exerting the weight mg which is downwards. Then the second is the contact force which is the has two component the N and the F. So, I am going to use notation or capital F for friction. So, this horizontal component which is so as shown in this figure, so this N is in the direction normal to the inclined plane and the horizontal component F is in the direction which is parallel to the inclined plane. And then as given in the problem, there is an external force which is also happens to be equal to Mg.



a Free Body Diagram (FBD)

a) Weight, Mg ↓

b) Ext. horizontal force Mg →

c) contact force
- Normal N
- Horizontal F (friction)

Σ Force = 0

So, if we draw the forces, the magnitude and the line of action and all these things. So, imagine this is the center of mass of the block. So, this is the weight acting downwards. The contact force has two component, normal component in the normal direction and the horizontal component in the horizontal direction. And then this represents the external applied force mg.

And in this case, our coordinate system is such that, so we are going to represent the direction parallel to the surface as let us say x-axis and perpendicular to the inclined plane as y-axis. And we are going to resolve the components of forces parallel to the plane and perpendicular plane as is easiest or natural as you have already you know. Then so once we have this free-body diagram, then we apply the condition of mechanical equilibrium that the all forces that are parallel to the surface, sum of all forces parallel to the surface must be 0. And sum of all forces perpendicular to the surface must also be 0. In this case, all the forces are passing through the same point which is the center of mass of the block.

Resolve components $\parallel$ to plane and $\perp$ to plane.

$$\parallel: \quad \underbrace{Mg\sin\theta}_{\text{weight}} - \underbrace{Mg\cos\theta}_{\text{applied for}} - F = 0 \quad \leftarrow \sum F_{\parallel} = 0$$

$$\perp: \quad N - \underbrace{Mg\cos\theta}_{\text{weight}} - \underbrace{Mg\sin\theta}_{\text{applied for}} = 0 \qquad \sum F_{\perp} = 0$$

So, we do not have to, so the torque due to all the forces are automatically balanced. So, we do not have to consider a separate torque balance. And applying this condition, we get equation. So, this mg sin θ which is coming from the weight and this Mg cosθ is coming from the applied force. So, this is weight, this is applied force.

And similarly, in the normal direction, we have the normal component and this Mg cosθ which is coming from the weight and this Mg sinθ which is coming from the applied force equal to 0. So, we solve them. So, this equation give us an expression for the frictional component. This equation gives a give us an expression for the normal component.

$$F = Mg\sin\theta - Mg\cos\theta$$
$$N = Mg\cos\theta + Mg\sin\theta$$

So, we have an equation for F and N. So, do you know the magnitude of F and N? Answer is yes. If we, if the θ is fixed, then this is the answer for F and N. But the question asks that what is the range of θ, which means θ can vary. Why can θ vary? Because the frictional force is kind of self-adjusting.

So, its magnitude, so Coulomb's law says this magnitude can be anything which is less than or equal to μ times N. So, that means if I plug it these two expression and combine it with Coulomb's law of friction, we get an inequality that and so to represent a magnitude, we use this modulus sign that the magnitude of friction is less than equal to the magnitude of the μ times the magnitude of the normal component the force. So, that gives a range for θ because for, so there is a upper range of θ such that if the frictional force varies from 0 to μN. That is why θ is not unique. There is a range over θ over which the the friction force, so the block will be at rest and the frictional force and normal force will be given by these expressions.

To solve these equations uniquely, we need more info about contact force.

$$\text{magnitude} \rightarrow |F| \leq \mu N \quad \leftarrow \text{inequality}$$

$$\boxed{\cancel{M}\cancel{g}\,|\sin\theta - \cos\theta| \leq \mu \cancel{M}\cancel{g}\,(\sin\theta + \cos\theta)}$$

So, then with this equation and note that the Mg gets cancelled on from both side from this equation. So, our answer the range of θ does not depend on M and g. It is kind of interesting to note. So now, we can simplify this further. So, this first we have to remove this magnitude.

Now, there are two case we will consider first. So, first let us assume that θ is less than π/2, the 90^0. So, in that case, the cos θ is positive. But still this when we remove this magnitude sin θ can still be greater than cos θ or less than cos θ. So, first let us consider the situation where tan θ is less than 1, which means sin θ is less than cos θ.

So, in this case our mod sin θ-cos θ is actually will be cos θ-sin θ which is less than equal to μ times (sin θ+cos θ).

$$\cancel{M}\cancel{g}\,|\sin\theta - \cos\theta| \leq \mu \cancel{M}\cancel{g}\,(\sin\theta + \cos\theta)$$

$$\therefore \cos\theta - \sin\theta \leq \mu(\sin\theta + \cos\theta)$$

$$1 - \tan\theta \leq \mu(\tan\theta + 1)$$

$$\tan\theta(1+\mu) \geq 1-\mu$$

Case 1:

$\tan\theta < 1, \cos\theta > 0$

$$\boxed{\tan\theta \geq \frac{1-\mu}{1+\mu}}$$

Similarly, in the other case when tan θ is greater than 1, in the same way you can show that we get a condition that

$$\boxed{\underline{\text{Case 2:}} \quad \tan\theta > 1, \cos\theta > 0 \qquad \tan\theta \leq \frac{1+\mu}{1-\mu}}$$

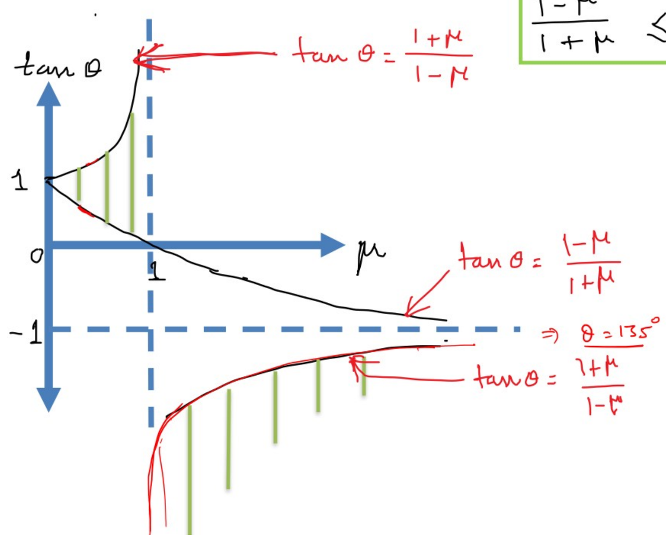
which happens to be the inverse of the previous ratio. So, these are equivalent to the condition that Coulomb's law that F is less than equal to μ times N. So, if we combine them together we get a double line equality that the θ should be such that

Putting together,
$$\boxed{\frac{1-\mu}{1+\mu} \leq \tan\theta \leq \frac{1+\mu}{1-\mu} \qquad \underline{\text{Ans.}}}$$

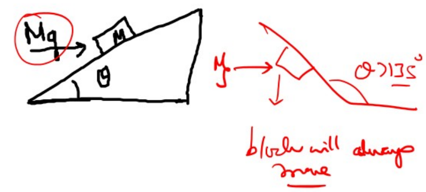
So, this is the answer. Now, so you can ask that okay so this is fine and we have seen this kind of problem before, what is the surprise? So, let us now realize that so we determine this θ in terms of μ but realize that in a given situation the θ and μ are independent.

So, what do I mean? So, you can vary the angle of inclination anything you can choose given a block and the μ which means because you know that μ depends on the nature of the surface contacts the material of the contact surfaces. So, let us say if a block is made of wood and inclined plane is made of let us say concrete then the μ is fixed. So, and then you can change the inclined plane. So, you can vary θ at constant keeping μ fixed and you can also do the opposite. Like you can keep θ fixed and vary μ by changing the material of the block or the material of the inclined plane.

So, θ and μ are independent parameter. So, then that kind of situation it is very useful to sort of explore what is called the parameter space that is you draw μ as your x axis and tan θ on the y axis. So, instead of θ we consider we use tan θ for mathematical simplicity because we know the expression for tan θ. Then we can draw these functions 1 minus μ by 1+ μ. So, this curve this black curve which is decaying represents the function 1-μ divided by 1+μ and this green curve represents the function 1+μ by 1- μ.



$$\boxed{\frac{1-\mu}{1+\mu} \leq \tan\theta \leq \frac{1+\mu}{1-\mu}}$$



So, first consider the case where the angle can be is less than 90^0. So, the angle of inclination varies from 0 to 90^0 less than 90^0. So, let us say now if we take μ equal to 0 which means a frictionless surface no friction is present. Then it gives you that there is only one value of inclined plane angle possible for which the given arrangement. So, for your reference I have shown the arrangement of the situation that you are analyzing.

Only for θ equal to 45^0 the block will be at rest because for any other value of θ the force will not balance in absence of friction and hence the block will start to move up or down. That is the first thing and I am sure that you may not have thought about that without looking at this it is not very clear when you are not looking at this kind of picture. Then suppose you are increasing μ from 0 to sort of start to make the surface frictional and gradually more and more friction. Then our inequality say that the range of θ is so there is the range of θ is shown by so the tan θ can be less is less than this curve upper curve and limited between the bound from the up above and below by these two curves. So, these are the possible ranges and what you can see is that as you increase μ then the range of allowed values of θ becomes bigger and bigger.

Then what happens when μ equal to 1 so then this inequality shows is that tan θ goes to infinity. tan θ is infinity when θ is π/2 that is a complete vertical case. So, that means any value of θ is possible from 0 to π/2 if μ is 1. One more even more interesting is the case if you consider the case when μ is bigger than 1 which means the friction is not dominating the normal component. In this case if you go back to the analysis this starting from this particular case in this case it will be easier to look at the it is easier to sort of start from the starting and take this kind of example and consider analyze the problem in terms of β which is π-θ and re rewriting the analysis.

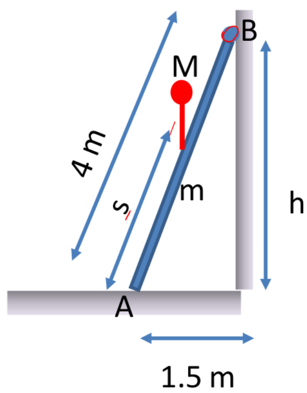
And in this case you will easily following the same route you can easily show that the inequality that will be valid is that tan θ is less than this 1 +μ by 1-μ . The other part of the inequality will not be relevant. The only case you need to consider is that when is this particular case. So, this means now when θ is bigger than 90^0 then θ and θ is a negative number. So, and you can immediately see that let us say if you take θ as something like little bit less above 90^0 than tan θ you can easily check is a huge negative number.

So, that represents this part. So, this curve actually goes so it diverges asymptotically goes to infinity. So, at when μ equal to 1 it again diverges. So, this is the region where I allowed values of θ. Now what happens when μ tending to infinity? What we see here that when μ tending to infinity then this curve approaches the value -1 which means tan θ is -1 which represents θ to be 135^0. So, this represents that no matter what is your coefficient of friction is, if you consider this kind of situation then the highest value of inclined plane possible for which this block will remain at rest no matter what friction coefficient you have is only 135^0.

If your angle of inclination is bigger than 135^0 then then this block will always move. So, I am sure that this is a kind of not at all obvious when you analyze the system. And, so this is a important useful trick that I want to sort of tell you is that when you have some parameter control parameter in your problem identify what are the control parameters are and then explore the full range of the control parameter it often leads to surprising inside. Now of course I must mention that if you take so this is true given this special value of this external applied force which is happens to be equal to the weight of the block if you change that if you take a applied force has a different magnitude. Of course, the conclusion the value highest value of the angle possible will be different.

So, now I hope that you got something surprising out of a very simple familiar situation. Now let us take another example which is sort of some simple way simple application of friction in day to day life. So, this is something called a ladder support. So, the friction between a ladder on and the floor. So, the problem is the following that there is a ladder AB whose length is 4 meter and mass is 15 kg and it is at rest on these two surfaces two walls and the floor and the wall as shown.

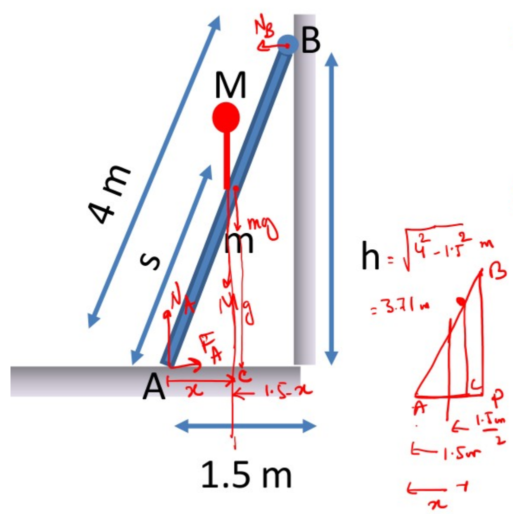
Practical application of friction



Determine the maximum distance *s* to which a *M=90 Kg* painter can climb without causing the *4 m* ladder *AB* to slip at its lower end *A*. The top of the *m=15 Kg* ladder *B* has a small roller. The coefficient of static friction between ladder and the ground is *0.25*.

A: s=2.55 m

So, those of you who have climbed the ladder, this is a familiar scenario and we want to compute this distance S. Now we are going to sort of give you so again we are going to use this force balance and torque balance approach and sort of quickly sort of go through the state and maybe the equations. So, let us take the ladder our system is the ladder then the surrounding is of course everything else which means the earth the floor and the wall. So, what are the interactions? So, the interactions so ladder+painter because we need to keep the painter in mind this is our system the combined system of m and M. So, the interactions on this combined system is the weight the force due to the earth.



Now where is this force going on? So, for simplicity we have represent this person by a line to indicate this is the line of action of the force so passing through this line passes through the center of mass of the person. So, let us say this is the center of mass then the force the weight of this person is something like mg and this is the so this is the line of action of the this weight. Let us say that this distance let us call that C where it hits the floor and let us say that this distance AC is x then by from the picture it is clear then that so this is x meter so let us say that this distance will be 1.5-x meter. The other interaction is the contact force between the floor and the ladder which in general has two component the normal component and the frictional component.

Now where is what is the direction of friction if the ladder sleeps then it will slip in this direction so the relative velocity is this direction so that means the frictional force must be in this direction which is opposing that from that to happen. Now the other contact is with at the point B and here the problem says that is a roller. So, significance of this roller is that there is no cannot it cannot support this contact cannot support a force in parallel to this plane it will start to roll it is a completely frictionless contact so the only force that is possible is let us call that in B and let us call that in A let us call that F A. So, these are the forces in the system. Now what about the weight by the note that the weight by of the ladder passes through its own center of mass and for simplicity so this is the Mg center of mass and this line of action and the line of action of the weight of the painter does not follow the same line in general.

So, if what is the line of action weight of the ladder so by by symmetry we can assume that this this center of mass is at the midpoint of the ladder. So, then if I draw this picture clearly so this is 1.5 meter this is the ladder then this is the midpoint and then by geometry it intersects this this floor at the midpoint between the distance from A let us call that P midpoint of B so A P is 1.5 meter so this point point must be 1.5 meter/2 and let us call that S. So this S is the line of action. So this point is C, so this AC is x.

So now we write down the condition of equilibrium so the force balance and torque balance. So first force balance let us take this as our x axis and this as our y axis. So let us consider the total force on the x direction must be 0 so, the sum of all force in x direction so there is a force on this combine system is

$$\sum F_x = F_A - N_B = 0 \rightarrow F_A = N_B.$$

$$\sum F_y = N_A - Mg - mg = 0$$

$$\rightarrow N_A = (M+m)g.$$

Similarly the total force on y direction is equal to 0 for force balance. So let us ask what are the unknowns in this case so the unknowns in this case are the three unknowns. So the contact force at A, N_A and F_A components of contact force N_A and F_A and contact components of contact force N_B at the contact B.

So you have two equations and three unknowns so we need one more equation that comes from the torque balance condition. Now note the torque balance condition so let us consider that we have our pivot point is B, so you are computing a torque around the point about the point B. So the line of actions of N_B passes to the pivot point B. So that will not contribute any torque. So, all other forces will contribute any torque. So you can check for yourself you can analyze this problem from other pivot points and you will arrive at the same answer. So, what is the torque due to the force A so this is effectively a 2D problem so the moment of force or torque is very simple is that the distance between the line of action from the point B perpendicular distance times the magnitude of the force and the direction of the torque is either perpendicular to the screen either going upwards or going downwards that you can determine from the right hand thumb rule.

$$\sum M_B = -N_A \cdot 1.5\,m + Mg.(1.5 - x)\,m + mg.\frac{1.5}{2}\,m. + F_A.h = 0$$

So, the contribution torque due to N_A is this N_A times 1.5 meter now the and this is in a clockwise direction let us call that negative the torque due to the weight of the painter is in the opposite direction so this is Mg times 1.5-x and the torque due to the weight of the ladder is Mg times 1.5 divided by 2 meter and the torque due to the friction is this height h so, F_A times h and this torque balancing so they are 0.

Now we can simplify it little bit so use this condition so that we get that now note that from the from the problem so h is known so h is from geometry this is 4^2 -1.5^2 and the unit is meter and this turns out to be 3.71 meter.

$$F_A.h - (\cancel{M} + m)g.1.5\,m + Mg(1.5 - x)\,m + mg\frac{1.5}{2}(m) = 0$$

So, let's simplify this further so let me write it here so we have

$$F_A\,h - mg.1.5 - Mg\underline{\underline{x}} + mg.\frac{1.5}{2} = 0$$

So then we have x which represents the line of the distance of this line of action of the center of mass of the painter. So we get an expression for x which is given by

$$x = \frac{\underline{\underline{F_A}}.h - mg.\frac{1.5}{2}}{Mg.}$$

So, this represents x now you can see that the mg and Mg so this term is a constant this term is a constant so x is basically depends on the F_A. So if more bigger the F_A the more away from x the this line of action. If F_A is small it is closer to this point of contact A and F_A is big means it is going further and further so as the painter climbs on the ladder so this point line of action the distance between the line of action of the same weight from this point contact point A becomes bigger and the frictional force is also bigger. So then but the painter cannot climb indefinitely the all the way till the contact point A because F_A is limited by this Coulomb's law which is given by this expression.

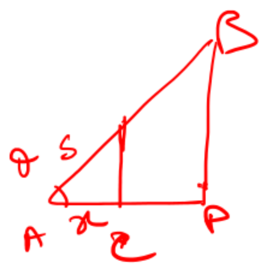
$$F_A \le \underline{\mu N_A}$$

$$\mu (M+m) g.$$

$$x \le \frac{\mu.(M+m)\cancel{g} - m\cancel{g}.0.75}{M\cancel{g}}$$

$$x = 0.957 \text{ m.}$$

And then we know that from our analysis that this is (M+m)g from force balance condition. Now this is x but what you want to know is S, so that we can get from geometry



$$\frac{S}{x} = \frac{AB}{AP} = \frac{4m}{1.5m}$$

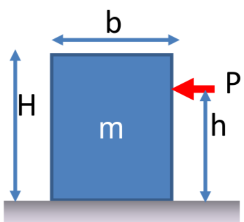
$$\Rightarrow S = \underline{2.55} \text{ m}$$

So, now this important point is that the this is the maximum value the painter climb can climb after that if he climbs or she climbs long higher then the ladder will start to slip. But this is an experimental demonstration that the painter position can be anything from from the floor up to this distance anywhere and if the painters position is lower than this value of 2.55 meter along the ladder that means that from this analysis we can see that it means that the frictional force is also less than this value of μ times N_A. So μ given because no matter you see from the force balance the N_A is constant no matter where the climb painter is climbing N_A is always remains the same μ is also constant.

So, μ times N_A is a constant value but that means that the friction but the friction is can vary so the fact that the painter can climb from up to a certain distance S not just one particular value of s but a range of distance it can cover on the climb from on the top. So, this is actually directly shows demonstrates that friction is a self adjusting force and it's vary value can magnitude can vary from 0 up to a certain maximum.

So now I quickly mention the another exercise for you to try.

Take home exercise: tipping over



The homogeneous rectangular block of mass m, width b, and height H is placed on the horizontal surface and subjected to a horizontal force P which moves the block along the surface with a constant velocity. The coefficient of kinetic friction between the block and the surface is μ_k.

Determine,
(a) the greatest value which h may have so that the block will slide without tipping over and
(b) if $h = H/2$, the location of a point C on the bottom face of the block through which the resultant of the friction and normal forces acts.

Ans: (a) $h = \mu_k \frac{b}{2}$ (b) $\mu_k \frac{H}{2}$ from mid point.

So, tipping over means it will go like this and suppose if H is the middle of the point of application of the force H is at precisely at the middle of the block what is the location of the point C on the bottom face of the block through which the resultant of the friction and normal force acts. So I leave you leave it as a take home exercise to for you to practice the free body diagram and to analyze the friction correctly. So, to summarize these problems are meant to illustrate you how to analyze the friction problems correctly so we have to remember that friction force is a self-adjusting force which can vary within a range and the μ can have any value between 0 to infinity. In the next lecture we are going to consider the friction between solid and the fluid that is the drag force. Thank you.