Course Name: Newtonian Mechanics With Examples
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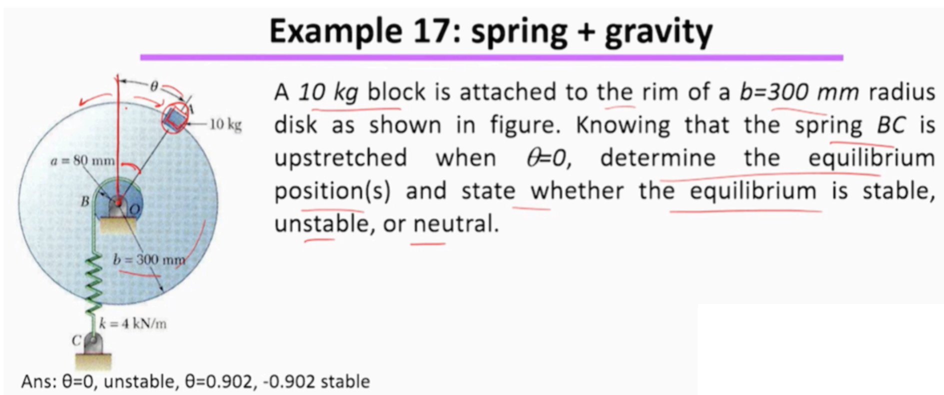
Lecture - 22

In our last lecture, we started discussing about the stability of a system in terms of its potential energy profile. So what we said was that a third way of looking at the system in mechanical equilibrium is that the condition of mechanical equilibrium is that the potential energy of the system is at an extremum. Which means either minimum or maximum. Let us understand this concept by taking an example. So this, when we sort of do this energy diagram method, so we are sort of, we are, for simplicity We are going to ignore all the friction present in the system because it is slightly conceptually difficult to define a potential energy in presence of friction, because friction is not a force that you can express as a derivative of an energy function. So here is an example.



So we have a mechanism, as shown in this figure. A 10 kg block is attached to the rim, so this is a 10 kg block attached to the rim of a disc. So this is a disc whose radius is B, which is equal to 300 mm. And at the center, this block is also attached to some wheel which can rotate, and this wheel is attached to a spring BC.

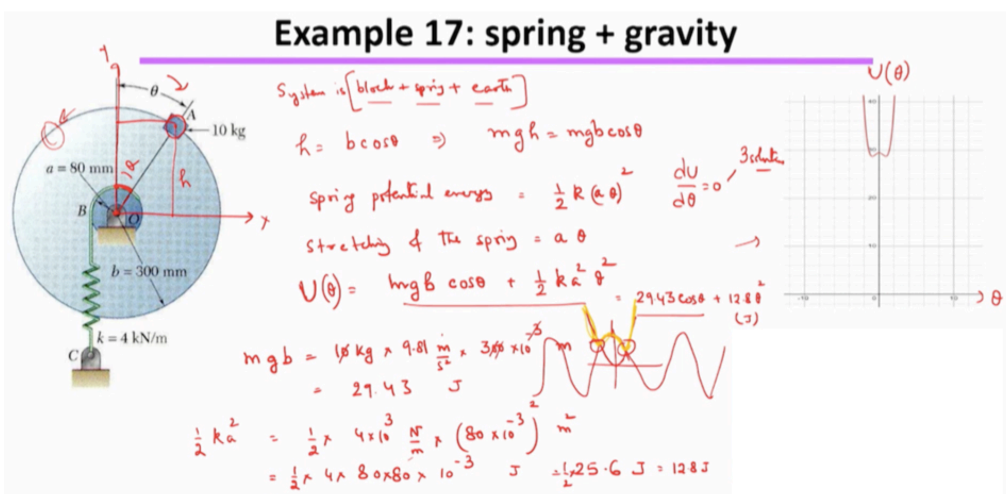
Now this is the mechanism. So if you rotate, if you move this block along the rim, then if you, so let us say at the vertical position is where the spring is unstretched. So if you move the block in this direction, then because it is attached to the way this mechanism is designed, then the spring is going to stretch and if you move the block from this direction, then the spring is going to compress and the block, if the block moves this way. Equivalently, if you compress the spring, if the spring compresses, then the block moves in the leftward direction and if the spring stretches, then the block moves in the right direction. This is the mechanism.

So the question is, determine the equilibrium positions and state whether the equilibrium is stable, unstable or neutral. So the first question is, how do we determine the position? So in this case, so what is our system? Our system is the block with a mass of 10 kg. So how do we determine its position? It is clear from this figure that it has a single degrees of freedom, even though it rotates on a disk which is a 2D object, but it is constrained to move along, so the constraint is such that the distance from the center of this point O to the block is always fixed, So it is a single degrees of freedom system. So we can represent this angle theta, so as a very independent variable. So each value of theta represents one position or one configuration of our block, and in the problem It is given, then when, so the theta, so this angle is theta, it is measured from this vertical direction, so this vertical direction represents the theta equal to 0 configuration position.

So I hope now you have some idea about the mechanism. So let us now go to analyze the problem. So what we are trying do? Our approach is the energy diagram approach. So, we are going to construct an energy diagram, which I have shown in this picture here. So this is the energy profile U, Where U is the total potential energy and this is our angle theta.

So we are going to write it down. Our goal is to determine the shape of this profile. So for that what are the potential energy? what are the forces? so instead of force. If you think in terms of energy you can easily see that this block, so let us say If I take this block plus this spring, so let us say our system is block plus spring. In fact in this case to properly define this potential energy we have to include the earth also in our system.

So this is our system, so that there is no external force acting on the on the system. Now there are two contribution to the energy. The first contribution is as you change theta the position of the block changes which means the gravitational potential energy of the block changes. So how much is the gravitational potential energy? Let us take this point O as our origin and this is our x axis and let us say This is our y axis, so theta equal to 0 is the y axis. Then, if I measure, if I take the 0 of the reference level of the gravitational potential at x axis, then its gravitational potential energy is given by the height of this block from the x axis, and from this picture, it is clear that this height is actually, so this angle is theta, so height is B cos theta.



So that means the gravitational potential energy is mg times h, which is mg B cos theta. So if h is positive, it is positive, if h is negative, it is negative. The second one is that as the block changes position, the spring can stretch or compress. So that gives you another source of potential energy, the spring potential energy. Now how much is the stretch of the spring? Suppose I start the block from a theta equal to 0 position and I move it to a position where it is at the currently.

So then it is clear from this picture that the point which was here on this small dial the green, then this stretch of the spring is given by this particular arc length which makes the same angle theta. So this is the stretching, so let us say it goes to the right, so it is the stretching of the spring. So the stretching of the spring is given by this particular arc length that I have highlighted here the radius of the smaller dial, which is a times the same angle theta, so this is from the mechanism. So note that if theta is negative, that means the block moves to the left. Then this actually is a compression.

So then, I am assuming that you are familiar with spring potential energy, So the spring potential energy will be given by half times the spring constant k times the stretching which is, then for our system of this block and spring and earth the total potential energy of this particular system is given by the two particular contributions you can have to add them. So let us now see that there is only one variable on the right-hand side and hence this is a profile of potential energy and you can actually, it is a profile with a single independent variable Theta, and you can draw them, and what I have shown on this picture Here is nothing but this particular function. So how do you guess what is the value of this function? What is the shape of this function? Well it is easy to guess, Note that it has two term, so the trick is to draw each term. So you know that, how cos theta look likes, all these things are constants. So, it will be easier if you sort of plug in the numerical values.

So let us calculate the value of Mg times b. So M is given to be 10 kg, let us assume g is 9.81 and b is the radius, so remember that we should express this radius; this is very important for numerical calculation. To be consistent, always try to use SI unit, so it is given in the millimeter, convert it into meter. So that is 300 millimeter which is, so and I also highlight deliberately these units, So what is going to be the unit of this quantity, Mg b? So Mg b has you expect; it has a dimension of energy, so you expect that it should be.

The unit should be joule in SI and you can verify that if you make this product, the unit The unit is indeed joule and if you sort of look at the numbers, this will cancel with this power, so you get 9.81 into 3, which is about 29.43, and this is kg meter square per second square, which is joule. Then this k is a square, k is 4 kilo Newton, kilo is 10 to the 3 Newton per meter times a square, which is 80 into 10 to the minus 3 whole square meter square. So again, you see that the unit is Newton times meter, so 1 meter gets cancelled.

It is Newton times meter, which is again joule. Now the numerical value, so what we have is half into 4 into 80 into 80, so there is a 10 to the power minus 6 coming from this square and 10 to the power plus 3, so you have a 10 to the power minus 3, and you have a. If you simplify it, you will get something like 25.6 joule, so this is 8 into 8 into 4 by 2, which is half into which is 12.

8 joule. So then this u theta is equal to 29.43 cos theta plus 12.8 theta square, and the unit is expressed is understood to be joule. So how does this function look like? So the way to do it is to sort of first plot each of them, so you see that you know how to, how the cos theta looks, so it varies from plus 1 to minus 1, so that means some constant times, cos theta will vary from plus a to minus a. So the cos theta will give you, so at theta equal to 0, cos theta is 1, so it has the maximum.

So, it looks like this; on the negative side, it looks like this. Now this something a theta square you know that this is an equation of parabola, the theta equal to 0; this is 0, at theta equal to, and then it grows. So then the total if you add them, so at theta equal to 0, it should start from here. If you add the two pieces, two terms. This is the second term, and this is the first term.

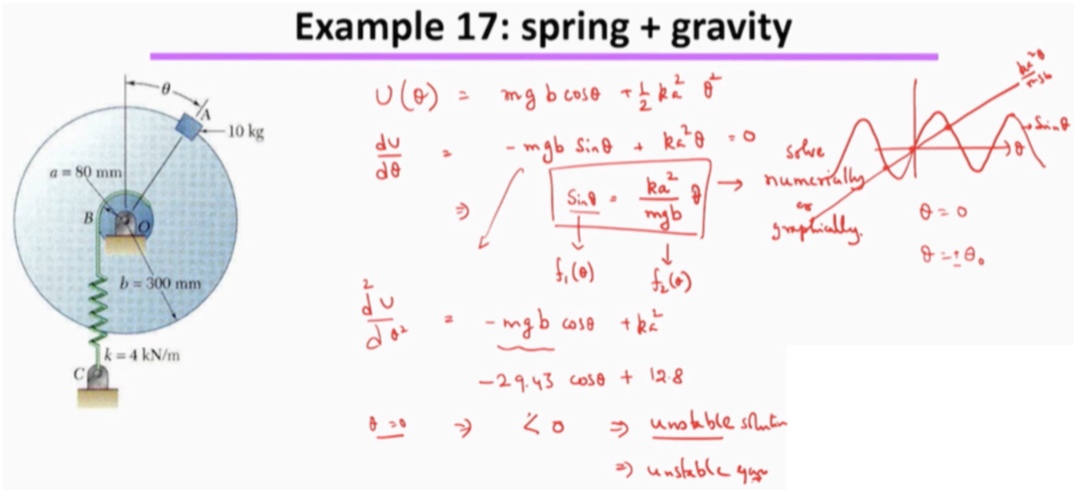
So the net force, net profile is going to be something like, now when start from here and it goes there and then it makes a turn because, for a small value of theta, this cos theta term dominates. This second term contributes little, but as you increase theta, larger and larger then the cos theta can never become bigger than 29.43, but the this second term does not have any restriction, so for large theta, this is going to be the dominating term, The spring term is going to be the dominating term. So we expect a net profile to look like this and indeed, if you draw this function, you actually see that. So now we are coming back to the question, so we have to determine the equilibrium positions.

Now this is easy; once you draw this profile, this is easy, as you can see from this profile or even if you look at shape of this curve, there should be three equilibrium position because there are two minima of u, two minima of u, and one maxima, one maxima. So there are three different values of theta. Where the d, the slope of u, the d u d theta is 0, three solutions. Now does it make sense? This is what mathematics gives as the answer, but does it make sense physically? Well, the first solution is theta equal to 0. We see a solution, so at theta equal to 0, the spring is unstretched, and this is sort of at the top position.

So you can see that the spring is unstretched means there is no contribution from the spring. But this block is here and this is sort of an unstable position, but it is kind of at equilibrium. Now what happens if you sort of the blocks gets disturbed slightly, now it sort of starts fall, starts to fall along the rim; let us say it falls on the right-hand side. Now, as it falls, the spring starts to sort of stretch and because of the stretching of the spring, the spring will try to compress so that it will naturally its tendency is to compress downwards, which means it is trying to give a force on the block in the opposite direction. So it is trying to prevent the fall of the block.

So then there will be some position at some finite value of theta, and there will be some position where you expect that these two effects will balance each other and that will be a equilibrium position. Similarly, on the other side, if it starts to fall on the left-hand side, Then the spring is going to compress, and the spring will not like it, so it will try to stretch itself. So again, the net effect is to give and try to prevent the fall of the block. So again, when this force will, initially the spring force will be small and as the angle theta increases then it becomes more and more stronger, so eventually it will balance this fall, and it will prevent or stop the fall of the block. So hence, you will get another position, so this is the left hand side position for negative theta, and the other one is this right-hand side position for positive theta.

Now, formally, since we have u is some mg B, let us write it in general terms, half k square theta square, then if I calculate the d u d theta, then we get minus mg B sin theta plus k A square theta and if we set it to 0, so that is the condition for extremum, maximum or minimum, then we get an equation that sin theta is equal to k A square divided by mg B times theta. So this is an equation. Now this equation is a non-trivial equation in the sense that it is, we have to solve it numerically. There are computational algorithms by which you can solve this kind of equation, It is a non-linear equation, or graphically, what do we mean by graphically? So you take this theta, so there are, consider this as some function 1 of theta and this is a second function 2 of theta. So the second function is like y equal to mx, so something constant times theta is y equal to mx.



Let us say this is the function k A square theta by mg B and the sin theta. You know that sin theta starts from 0, and this is sin theta. So, as you can see, there are three points of intersection. So this point of intersection represents the fact that the value of sin theta is equal to the value of this function; this is sin theta. So these are the points which satisfy the values of theta, which satisfies this particular equation and You can see from this graph that there are only three solutions possible.

Now if you want to know whether this is stable or unstable, it is clear from this graph or from looking at this equation, theta equal to 0 is a solution and the other two solution are some particular values of theta naught and by symmetry, you can see If theta naught is a solution, then minus theta naught is also a solution, so they must be symmetric. Now let us calculate the second derivative. So the second derivative from this equation is minus mg B cos theta plus k A square. Now, from the given, if I plug in the values, we get 29.

43 cos theta plus 12.8. So if I take, so this is minus, so if I take the theta equal to 0, we can check, It is easy to check for theta equal to 0, which is negative, which means this is an unstable solution. So unstable solution, which means that unstable equilibrium. So I hope that this gives you some idea about how to approach this kind of problem. To solve this kind of problem. So here I want to mention quickly some interesting applications of this concept.

Remember that in the last lecture we said that the energy diagram method of doing stability analysis. Which is useful to sort of understand the question: What happens if you move away from the stable point? and How many equilibrium configurations are possible? But there are various interesting examples beyond mechanics. For example, let us start with something in the mechanics. So we can use this energy diagram, It is a very powerful method, and we can use it to determine pictorially the bound states and unbound states of a system. So a famous example is the two body system, for example, motion of earth around the sun or motion of an electron around the nucleus in a hydrogen atom, etc.

So let us say that R, which represents the distance between two objects and the configuration of our system. Which consists of two point particles, are specified completely by, are specified by, uniquely by R. And then we have this: for each R there is some potential system, has some potential energy. And this red line, sorry, the blue line-represents this potential energy profile as a function of different positions R. Now the system is moving, so it also has a kinetic energy in general.

So the total energy of the system is given by the kinetic energy plus the potential energy. So these green lines are the total potential energy of the system. Now we have shown you three different values of the total energy of the system. Now, if you do the analysis for that, turns out that if you just look at this total energy, You see that this is cutting this potential energy diagram profile at two points. And this means, and we are, so this is, that these are in a system where there is no friction, so the total energy is constant.

So it immediately tells you that the distance, this R, values of R can only vary within this range. So the distance between the two objects cannot be less than this particular value and cannot be bigger than this particular value. And the system is stable. This kind of state is called a bound state. Now if I go to slightly increase the total energy of the system, then we see that the range of values, Over which the separation, the R can take the possible values of, allowed values of R have increased but are still bound between a maximum and a minimum.

Now contrast this with the situation when the total energy is at level C. In this case, you see that it is, the R has a minimum distance, so it cannot go beyond this distance, But on the other side there is no restriction on the highest value possible for R, the separation. It can be any separation is possible. So this kind of state is a unbound state. So just while making this potential energy diagram, we can, it is very easy to understand the nature of the motion of your system in a qualitative way.

For example, this means that the, if you, if you consider the Earth and Sun system, it is basically means that the Sun is, the motion of Earth is bound to move around the sun. Its distance between the Sun and Earth varies, but it varies within the range, it cannot be anything. Whereas this C position can, for example, perhaps represents the motion of a comet. So it also moves around the Sun, but at a distance, so it can have the closest approach to the sun.

But there is no limit on the furthest approach. So the motion of the comet around the sun is not bound. Another example is from chemistry, where the chemical reaction is, so the why, How the chemical reaction happens and the mechanism of the chemical reaction. So there is a sort of thing thought of, understood as a transition from one state to another. So the initial state is the state of the energy state of the reactant components, and the final state is that of the product. And here, so the initial, it is also, reactants were not kind of stable, So, they are not doing anything and then when they come in contact with each other, sort of mix them together.

Then they do the chemical reaction and they go to a second state. So this kind of, describing this kind of chemical reaction, This potential energy diagram is a very easy way to sort of see how the reaction is moving from start to finish. The second, third example is that things become really interesting when, So far in our examples, the system configuration of state or position is completely specified by one or a single degree of freedom. But things become really interesting when your number of independent variables becomes more than one. So in this case, the potential energy is a function of more than one independent variables.

And so, hence, it is called an energy surface or if the number of variables is 2, and if the number of variables is 3, then it is called an energy landscape. So it is not important to know what are the details of this picture are. but this is showing two independent variables, one is plotted on the x axis and one plotted on the y axis. and these different colours represents the values of energy. So the grey has the highest value of energy and blue has the lowest value of energy, and These lines are sort of contours of equal energy.

So it is sort of a height map of a, as you see, so the blue is the deepest part of the region. So for example, if the green is higher, the energy is higher, and the grey, sorry the red, the energy is even higher. So this is like a kind of cross-section. So if you look at a kind of well, like this kind of shape, Then, if you sort of do a cross-section, then what is the depth of that part? So this kind of map is also common in, let us say, your atlas in various map in the context of geology. So where you sort of try to know what the depth of an ocean floor or the height of a land, mountain from sea level, it is the same thing.

And immediately, while looking at this curve, you look at what is the region where the energy is minimum. So the energy is at its minimum with the position of the most stable configuration. So you can see, just by looking at it, you can easily identify the, visually the region with most stable configuration. Now, even more generally nowadays, some of you will be interested in going to machine learning. So in different problems that involve machine learning, which are kind of technically called optimization problems.

So we can even generalize from the potential energy to something called a cost function. So this concept of cost function or fitness function is kind of the same kind that can be used as a function that is used to find the optimum state of a system. And this is a very generic function because it is used beyond physics. So it is used, for example, in curve fitting, in various machine learning problems, in economics, in stock market analysis, in biology, in analyzing the evolution theory, the proteins, the functions of proteins, drug discovery, etc., So, usually in those cases, the number of independent variables to define the cost functions is usually very, very high and hence, so the surface, instead of a simple 1D function or 2D surface, you have a very complex function.

The most important feature that you should remember of this kind of complex function at high dimension is that there is always, no matter what the details of your system are. There are always multiple equilibrium configurations possible, more than one minimum and maximum. So this is the most important feature of this complex function. So this is something that is beyond the scope of this mechanics course. I am sure you are going to learn them in different other courses about these things.

I just mentioned here, so that, because it is essentially the same concept of energy, the kind of generalized version of the potential energy, you have some configuration or state of the system from which you compute a number, which you call the cost function and then You have a profile of the cost function and for some values, Some configurations are optimum in the sense that the cost function is minimum. So the potential energy, or the cost function, is at its minimum. And then the problem is about finding those states. So with this, we finish our discussion on mechanical equilibrium. So in the next lecture, we are going to look at a very important force, which is the friction force. Thank you.