**Course Name: Newtonian Mechanics With Examples**
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**Week 05**
**Lecture - 21**

Last week, in the last few lectures, we discussed the concept of a dynamic system. We discussed the concept of constraint motion and introduced the degrees of freedom. stated the principle of virtual work and illustrated it with an example. So, for the sake of completeness, today I am going to first show to you that the principle of virtual work is equivalent to our earlier condition that we used that for mechanical equilibrium. That is, the total force is 0 and the total torque is 0. So, now I want to note that the word equivalent has a special meaning.

So, A equivalent to B means if I start from A. I can logically go to B, and at the same time, if I start from B, I can logically go to A. So, this is the meaning of equivalence. So, I am going to show you in two parts.

First, we will consider the case of a single particle or a single rigid body. which we can represent by a point object, and then we in the next part, we will take an extended object. Let's consider a point particle. Let us say a point charge placed at the centre. Now, there are other charges.

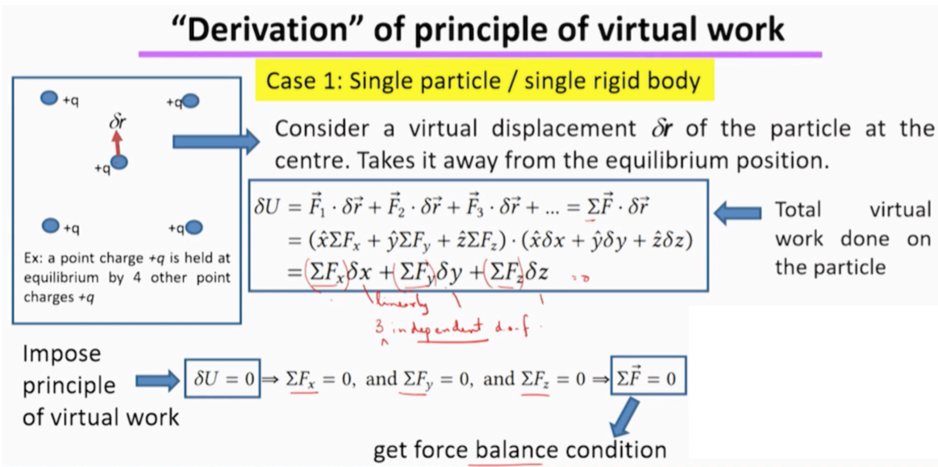
There are four other charges at the corner of a square whose, at the centre of the square, this charge Q is placed. So, because of this other charges. You can see that all the charges are positive. So, the total force from this charge on the centre charge and discharge on the centre charge is equal and opposite. They will cancel, and similarly for these two charges.

So, this is an equilibrium; the centre charge is at equilibrium, and the total force is 0. So now, consider a virtual displacement delta R. So, note that delta R is a vector. So any direction you move this charge a little bit in, let us say in this direction as shown by this little arrow. So, this is a displacement and you are imagining this displacement, So virtual displacement.

So, then it goes to a new position and in this position, This charge will be closer to the upper two charges, so the total force is no longer 0. So, it is no longer in the equilibrium position. So, let us calculate the virtual work. So, here, so the virtual work in this case. So in general, if there are, so in this is a case where this point particle is experiencing four different forces F1, F2, F3, and F4.

So, in general there could be more than four or less than four, so there could be n different forces on an object. So, the total work done by force, one is F1 times the virtual displacement delta R, and similarly for other forces. So, the total work is this sigma; this sigma represents the total, so this means the same force is sum. sum over all the forces and since the displacement is the same particle, so for all these works, the displacement part is same; you can take that all this part is common. So the total virtual work is given by the total force on the particle and the dot product with the virtual displacement.

Now we can sort of take any arbitrary Cartesian coordinate system and write down the total component of total force in the Cartesian coordinate system and similarly component of the virtual displacement. So, after simplification and applying the dot product rule, we get this component that the total virtual work is the total force in x direction time, the component of virtual displacement in the x direction, plus the corresponding term in the y direction and in the z direction. So, now, this is the relationship between the virtual work. So this is work or virtual work, and this is written in terms of the total force. So, now we have an expression for the total virtual work done on the particle.



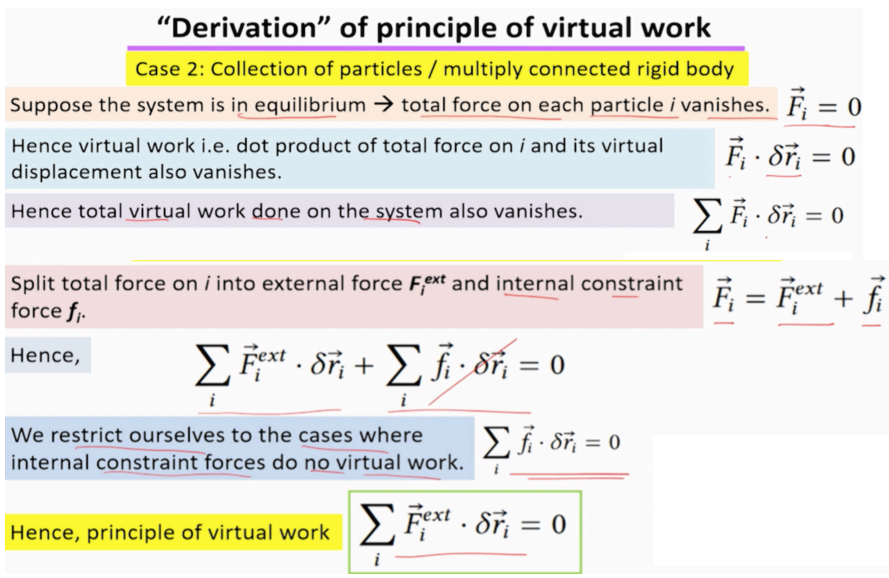

Now, if I impose a force balance condition, that is, you demand that in this equilibrium position the total force is 0. So, that the force is a vector, note that it is a vector, Which means that each of its x component, y component and z component all of them must be independently 0, which means the total force on x component is 0, and the total force on y component is 0. The total force z component is 0. And then if you put this 0, then this is what you can see that right-hand side of this expression gives you 0, that is, you get the principle of virtual work that if you give a virtual displacement, then the condition of equilibrium is that Total virtual work must be 0. Note that we are this: F 1, F 2, F 3, etcetera; these are all the external forces that we only need to from this calculation it is clear that We only need to consider the forces that do virtual work.

Now, if I do the other thing, that is, you impose that delta u is 0, that is, you impose the principle of virtual work. Then we get the equation that this sigma fx times delta x plus sum over total force in y direction times delta y plus sum over total force in z direction times delta z, which is equal to 0. Now, for a single-point particle this x, y and z. These are three independent degrees of freedom. So, there is no other constraint on the motion of this particles.

As you can see, for example, in this particular example, so you can see We can move this center particle independently in the x direction, y direction, and z direction. Now, if you have these three independent variable and their sum is 0, then, to be precise, They are linearly independent, then it is clear that this coefficient of each of the only way you can have three independent piece going to 0, only way it is possible if the coefficient of all of them are equal to 0. That means it follows that the total force of the x component, y component, and z component must be 0, which means the total force is 0. Hence, we get the force balance condition. So, in this case, there is no separate torque balance condition is not required.

So, this shows that we can start from the principle of virtual work and go to the force balance condition of mechanical equilibrium and we can start from the force balance condition of mechanical equilibrium and go to the principle of virtual work. So, this shows that they are equivalent. Now, if we have a collection of particles, what happens? or a rigid body which is multi-connected. so you cannot describe them as a single-point particle; you have to consider them as an extended object. Suppose, the system is in equilibrium that means total force on each particle i is 0.

So, Fi is the total force on each particle. So, if total force on each particle is 0 on particle i is 0, then if I take a dot product which is virtual displacement. So delta ri is a consider you giving virtual displacements of each particle or each component of the system. So, if you consider only the particle i, then the since Some Fi is 0, then the dot product of Fi times the virtual displacement of the particle i is also 0. So, if this is true for each particle in the system, then the total virtual work done on the system of taking all the particles into account must also be 0.




Now this total force Fi you split into two parts, one of which is external force. Which are not constraint forces, and the small Fi, which are the constraint force. Now, internal or the constraint force. Hence, whatever this from our previous equation, What we have is that this total virtual work is now split into two terms. One term is the contribution from the external force, and the other term is the contribution from the internal constraint force.

And now we restrict ourselves to the cases where internal constraint forces do no virtual work, no virtual work. So, they are just there to and so remember the discussion how is it possible. So, this is the condition. So, we have already discussed how different ways This condition is satisfied in a real situation and also I mentioned that We are ignoring sliding friction, which does work, but we are ignoring all those frictional work. So, if this term goes away, then it follows that the total virtual work done by all the external forces on the total system, which is given by this particular expression, must be 0, which is our principle of virtual work.

Now, before going to the next topic of discussion, let me point out one interesting sort of application of the concept of degrees of freedom, which is intimately related to analyzing this principle of analyzing mechanics problem with principle of virtual work, but this is not the only concept of importance of degrees of freedom. So, this is gives you lot of insights about the different topics. That you are going to study in your respective branches of science or engineering. So, in modern times, there is an emphasis of describing the topics that you are going to study not as a traditional way of physics, chemistry, biology, civil engineering, electrical, etcetera. But to look at it from a complex system perspective,.

So, what is a complex system? So, the way to understand complex system is to count the number of degrees of freedom. So, how many number of variables you need to specify one state of your system? So, the more number of variables you need, the more complex is your system. So, the more system of study. The other aspect of your of the complexity is the nature of the forces acting in the system, that is the interaction between the system and its surroundings. So, sometimes the equations that describe those interactions for example, in this course, The central equation that we are using to describe our doing our analysis is the Newton's laws of motion.

So, this there are two possibilities this equations the fundamental equations that govern the behavior of the system, they can be linear. For example, Newton's law or Let us say those of you who are studying electricity magnetism Maxwell's equations of electricity and magnetism can be non-linear. So, these are the two aspects. Using these two aspects, we can sort of organize a lot of the topics that you are going to study in your respective curriculum, this is a map and that gives you a lot of insight. a new way of looking at your syllabus.

So, for example, on the x axis, here is the number of variables. So, we can start simpler, and the y axis is the non-linearity of the interaction or the basic laws of laws governing the behavior. So, the simplest systems are one degrees of freedom, and some examples are here. So, these are like topics like RC circuits or, in nuclear physics the law of radioactivity. So, the system so, the easy part so, the second, then you can go to a slightly more complex example with two degrees of freedom.
So, this involves a very important phenomenon called oscillation and different kinds of oscillators, which are very important in physics, biology, chemistry, electrical engineering, and so on. So, they have linear oscillators and different spring mass system, or RLC circuit, Also the famous Kepler's laws of planetary motion, the movement of the earth going around the sun. And then you can keep on increasing the number of variables and you can get different types of problems, such as in civil or electrical engineering and so on. And then, when the number of degrees of freedom is very high, you get lot of very interesting natural phenomena Which are called collective phenomena. And here are some examples, like coupled oscillation or different properties of solids and something called in physics later, you will study when you have large number of degrees of freedom there is a branch of physics called statistical mechanics.

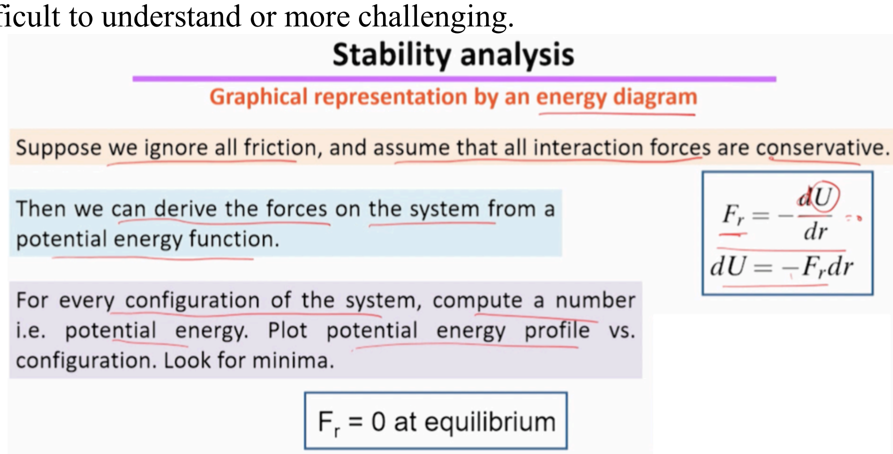
And finally, your degrees of freedom can become infinity, which means you are no longer so you are treating the system as sort of a continuous body and there you study from with that approach, the various wave phenomena such as electricity, electromagnetic wave or elastic wave or water wave, heat wave, sound wave etcetera. And similarly, so these are all you are just number of variables are increasing But in some sense, they are simple, simpler system because your basic equations of motions are always linear. So, they are mathematically simpler to analyze. Now you can go to the other axis and you can sort of add make the underlying equations more non-linear.

There will be difficult to analyze. So, what is the point of this sort of map? So, this sort of map sort of organizes your various examples from simple to complex. So, in this part, you have a linear equation of motion and a small number of degrees of freedom, such as 1, 2, 3, etcetera. So, this is kind of a simple example. As you increase the number of variables, The other part, which is simple, is this linear equation of motion and continuum. So, the theory of this kind of phenomena is very simple.

I mean, we can sort of describe because they can now be described by partial differential equations, which we know how to mathematically analyze.  So, this is also some sort of simple region. Now if you go from linear to non-linear, you add non-linearity in your basic laws of motion, then you encounter some systems, which are like in encounter lot of interesting behaviors such as chaos. And, for example, maybe a more familiar example is the pendulum. when you have a linear system actual pendulum is a very interesting system with lot of interesting motion possible.

But when you sort of think of it as a simple harmonic pendulum or simple pendulum you restrict yourself to the linear regime which is like so you assume that your, The amplitude of the oscillation is small. Whereas linear is an actual pendulum, if you remove the restriction so that the amplitude of oscillation can be anything. So, it can be even let us say 180 degrees. If you have the real pendulum, then this is an harmonic oscillation. You can get a lot of different behaviors like that.

Now the final thing I want to mention is that in modern times, the real challenging areas of science and engineering are what is called is so called frontier. So, this is the region where in this map where you have both the high lot of large number of degrees of freedom. So there are lot of phenomenon I will mention only two examples one is earthquake and another is turbulent flow in liquid and the third is the living life itself the living systems. So, if you want to study this kind of topic, this is the most challenging and more difficult to address because they involve large number of degrees of freedom and their basic governing laws are also non-linear. So, using this concept of degrees of freedom, you can sort of understand Why are some things easy to understand and some things are difficult to understand or more challenging.

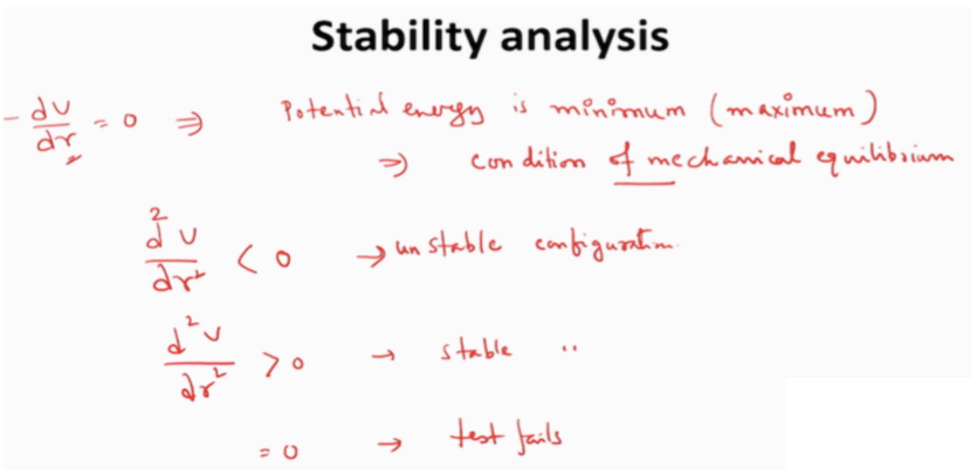



So, now we are going to start a sort of one more way to look at the  final way to look at the discussion of mechanical equilibrium. So far, we have sort of looked at the system in a configuration which is at equilibrium. But now we want to know the stability of the equilibrium.

So, there are two things possible. So, let me give you an example this is actually not a very difficult concept is very simple thing to state in mechanics.

However, this becomes important because nowadays this simple concept of stability finds applications beyond the normal traditional areas of mechanics and that is why I wanted to sort of give you exposure to this topic. So, let us see what is meant by stability. So, let us say that I have some kind of object and  I am going to imagine that this is a table. So, I am going to place this object on this table.

So, this is a configuration. So, if I keep it and this is in a mechanical equilibrium condition because if you for example, draw the free body diagram of this object you see that the total force will be 0. The total force due to the gravity on this bottle of water and the force by the table on the bottle of water, they balance each other. So, the total force is 0, total torque is 0. So, the system is in equilibrium. You can also do the same as I discussed today that using the principle of virtual work.




But this is not the only configuration that is impossible. There is you know that this is another configuration and This configuration is also in mechanical equilibrium. So, then you can ask that which of this configuration is more stable and we know from experience the answer is that this is more stable. Why you can ask? So, this is from experience and those of you who are familiar like kind of anticipate where am I going. So, you can say that okay this is more stable because in this configuration the centre of mass is lower compared to this position.

In this position the centre of mass of this bottle of water Let us assume that this is at the middle of the bottle. Then this middle of the bottle is at something like this height, whereas in this case this is at lower height. So, because the system can lower its centre of mass, which means that it can lower its Let us say gravitational potential energy. so this configuration must be more stable. Now I can ask another question which is that what happens? So, let us say I take this configuration if I disturb it from the equilibrium.

Now this is not a virtual disturbance, displacement is a real displacement. If I give a real displacement, what happens? So, there are two possibilities. One possibility is that it can return to its initial or original position. So, if I do this it will come back, if I do this it will come back, if I do this it will come back.

The other possibility is that it may go away. For example, in this particular case, Let us say instead of bottle if I take this pen and let us say if I put this pen on this table in this position. Now you may imagine that if I remove the pen now, it is possible. You can imagine that, in principle, the total force on this pen if I draw the free body diagram of this pen, then the total force on this pen in this position can be 0. The force by the gravity and by the table on this pen can cancel each other.

But if I remove it What happens is that the pen falls down. Why? Because in this case, Because the tip of the pen is so narrow, it is very difficult to achieve the torque balance condition. So, if I make a disturbance away from this particular configuration in that case the it sort of goes away. So, It goes away from this equilibrium position. So, there are these two possibilities. So, if the system return to its original equilibrium position, then that equilibrium is stable and If it does not return to this equilibrium position, then it is unstable.

But now there is a caveat to it here: if I take this bottle on this, now you know that this bottle know that there are two more than one equilibrium position. So, if I sort of disturb it for So, if I sort of disturb it with a small disturbance away from the equilibrium position, it will return. But if I give a large displacement disturbance away from the equilibrium position, it will not return. So, here is what we are when we sort of talk about stability: We assume that our disturbance and the displacement that we are giving are small.

So, this is a small displacement. So, this is very simple everyday experience. Now, we want to use this. This simple everyday observation to sort of construct some function that sort of describes a kind of so that we can sort of analyze the stability in a more systematic, formal way. So, here is a very powerful concept of an energy diagram. So, suppose we ignore all the friction that is there so for the time being, we imagine there is no friction and we assume that all the forces are conservative that is why we are ignoring friction.

Now, what do I mean by conservative? It means that we can derive the forces. We can express and write down the forces acting on the system as a derivative of an energy function or a potential energy function. So, let us say there is a force F acting on the system and then, basically, mathematically, we can express this force F as a derivative or a total derivative of some function u which we call the potential energy of the system. And this potential energy can have different sources, as we will see in the example. Now, for every possible configuration of the system, compute the potential energy, which is a number.

Which is a scalar because you see that this is a dot product so between the force and the displacement, so it must be a scalar. So, you get a number for every configuration. So, which means ,what do I mean by every configuration? So, if I take this bottle again and This is a configuration for which we can compute the potential energy. If we distribute or disturb It little bit further, this is a different configuration and again, we get a potential energy.

If we do this, I will get a different configuration. So, as I move this away, for every value of the position of this bottle or angular orientation of the bottle, we can compute a potential energy a number. And then we have a plot of the potential energy profile for different configurations. Then if the total force is 0 if this is 0 that is the condition of mechanical equilibrium. So basically, this means that the derivative of this potential is 0.

So, then the condition of potential energy minimum. basically means that the potential energy is minimum or it can be maximum. We have only looked at the first derivative, and this is a condition, so this is a third way to look at the condition of mechanical equilibrium. Now for the sake of completeness, I must mention that, as you know that the first derivative 0 does not tell you whether just tells you that the potential energy is minimum total force is 0 but what happens if you sort of disturb it slightly? That really comes from the second derivative of the potential energy, where R is the variable that describes the describes the configuration. Now as you know that if the derivative is 0 negative the second derivative is negative then this is a stable configuration.

Sorry, this is an unstable configuration. If this is positive then this is a stable configuration. So, this is a stable configuration. And if this is 0, then this test fails and we need to check it in different ways in general. So, in the next lecture, we shall illustrate this with an example. Thank you.