Newtonian Mechanics With Examples

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Week -04
Lecture -20

In the last couple of lectures, we introduced three concepts, the concept of constraint motion, then degrees of freedom and third the idea that the constraint forces do no work, ignoring the sliding friction. With this background, now we are ready to introduce what is known as the principle of virtual work or rather the d'Alembert Lagrange principle of virtual work. Now you know that this course is called Newtonian mechanics, but this concept was, as far as we know was developed after Newton. So, this came from these two gentlemen, d'Alembert and Joseph Lagrange from France. So, let us go through step by step through the principle of virtual work. First, you have a system.

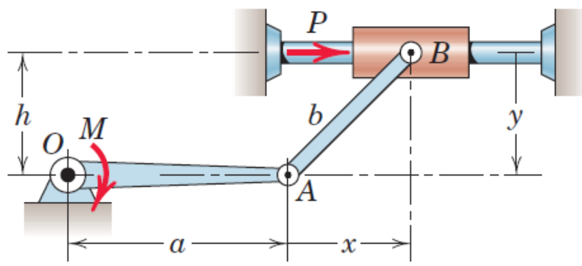
So, your system could be one particle, a collection of many particles, a rigid body, a single rigid body or a multiply connected rigid body, which means that like think of some machine parts, so different parts of a machine linked together. For example, if I take my arm, if I don't consider this as my system, this is, to some approximation is a single rigid body, but if I take from here to here, it is connected through a joint at the elbow, so this may be called a multiply connected rigid body. So, you have a configuration, so some configuration like this is a configuration, this is a configuration, this is a configuration. Now given this configuration, imagine, so it is an imagination, a infinitesimal that is very small, infinitesimal in the sense of calculus, very-very small tending to 0, virtual that is imaginary displacement to the configuration.

And you are allowed to consider only those displacements that are consistent with the constraints present in the system. Then we calculate virtual work δU done by all the external applied forces that is those forces that do work. So, what is a virtual work? It is same as any other work, so you know that this work is defined by the dot product between the force acting, let us say if I take a particle, the force acting on a particle times the displacement of the particle. This is the definition of work as you must know from your high school physics course. So this, what is the difference between what you have learnt before and the virtual work? Nothing, only that this displacement is not real, it is a virtual displacement.

So, you are imagining this displacement. Now this is the principle that the rule is the following: If the system, so we are going to state the condition of equilibrlabelium in terms of this virtual work, that is all. Earlier we stated it in terms of the force and torque balance, now we are going to state in terms of this work. If the system is in equilibrium, then the total virtual work should be 0 and vice versa, which means that if you calculate and find that the total virtual work is 0, then the system is in equilibrium. So, when you go both way, one to another and opposite, call them equivalent.

So, the condition of mechanical equilibrium, so now you can have another way to think about this mechanical equilibrium, why the object is motionless, not moving, because calculate the total virtual work and it is 0.

So, let us understand this through an example. So, here is an example, it is a kind of interesting mechanism, it is called a crank shaft slider mechanism. So, this is a picture. So, what we have, so basically this is a common machine part present in various kinds of devices such as the car and many things. So, if you want to, so here is a picture, I am going to explain the picture and I have put a link to a YouTube video where it show how to build such a mechanism yourself and it will also show you the how it moves.



DIY crank slider model https://www.youtube.com/watchv=kfLg2EmP6mM

So, basically there is a pivot point O and then there is this point A, it can rotate in a circle. So, this point A can rotate in a circle. Now this point B is connected to point A through this rod and if the point A and this point B is constrained, so it can slide on this channel. So if the, so this is a constraint that this is a slider, the B is a slider.

So, it can only move in one direction, it is not allowed to move in a circle, it only move along this line, this is the constraint. So, if point A rotates in a circle, then point P will, so if point A goes down, then point B will move this way, if point P moves up, then the point P moving this way. So, this is the motion of the mechanism. Now what is the question? So for link OA, so OA is a link in the horizontal position shown, now this is the collar, sliding collar, so this B is a sliding collar. Determine the force P on the sliding collar which will prevent OA from rotating under the action of the couple M, neglect the mass of the moving parts.

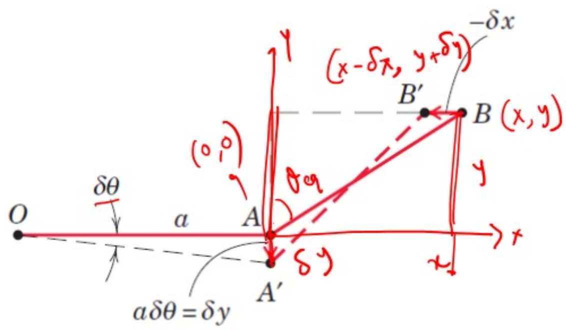
So, let us understand the question. So, there is a torque applied on this rod OA at point O, so that is trying to rotate this OA. So, if it can rotate, if the A can rotate, then the B will slide to the left and there is a force, extra force, external force is applied on the slider B to prevent that movement. The question is how much force you need to apply on the slider B to keep it at rest, keep it motionless. So, in this kind of problem, how do we go about solving this problem? So, what we do, so is that first you try to understand the actual motion.

So, first you identify which are the movable part in your mechanism. So, because most of this stuff are kind of, so it's an example of a multiply labelconnected rigid body and in this case, so first thing is to understand which part is moving and which part is not. So, here this point O is fixed, it is not moving. So, this rod OA is movable, so this point O can move and the point B can also move. So, this slider can move, so this is a moving part and this is a moving part and it moves in a circle, this moves in a horizontal line, simple.

So, the next thing is to understand the constraint applied on the motion. So, the constraints are the following. So, this is a rod which we assume as rigid rod, its length is fixed, which means the distance from O to A is fixed. So, when it moves in a circle, the distance between O and A does not change. Similarly, when A moves as you can see in this figure, so A moves, then B also moves, but the distance from A to B remains same because it represents the distance of this link rod which is small b.

So, this remains same. So, this is the constraint. So, this is a typical configuration of a system. So, let us think of this as a equilibrium configuration. So, let us write this.

So, this is an equilibrium configuration as shown in this figure. Let us redraw this. So, let us say that this angle is some θ equilibrium and then we now consider an imaginary displacement that is consistent with this constraint. So, what is the imaginary displacement? Let us imagine that we take a new configuration of the mechanism in which the position of this point A goes down a little bit and this is little bit. This will be very important. This is little bit. So, this goes down by an, so it rotates by an angle δθ about the fixed point O. So, it goes down here. So, then because these two parts are connected, it also means that this point B will automatically move towards A. So this is our displacement. So, this goes to a new configuration.



Now we want to understand the constraint. The easiest way to understand the constraint is to look at, suppose I want to find the position of this point B. Now you can choose any origin, like you have to define your coordinate system and you can choose anything. So, in this example, let us, I am going to choose the point A as my origin, and I want to measure the distance from point A. So, in that case, so this is my origin and then this distance is x.

So, this is my x axis, this is my y axis, and this distance is y. Then the distance between the two points, the reason I choose A, B because then the distance between this point A and B is nothing but the coordinates of the point B because this is origin, its coordinates are 0. And this is constant because of the rigidity of the rod and this is the constraint. So, we can write that-

$$x^2 + y^2 = b^2 = \text{constant}.$$

Now what is the coordinate of B'? So, the B' it is clear that it has a coordinate x-δx because it will have, its x coordinate is little less which is x-δx .

But note at the same time its y coordinate also changes because now we are measuring the if the point A moves, so then the distance from the point A, the new distance is from, is y+δy. So, then the distance between the point A' to B' is again

$$(x-\delta x)^2 + (y+\delta y)^2 = b^2$$

If I equate them, and we ignore the second order differential, the square of the differential and this is why you need infinitesimal displacement so that the displacement you always considered only in the first order in the displacement.

$$(x-\delta x)^2 + (y+\delta y)^2 = b^2$$

$$x^2 + y^2 = x^2 - 2\delta x \cdot x + (\delta x)^2 + y^2 + 2y \cdot \delta y + \delta y^2$$

$$-2\delta x \cdot x + 2y\,\delta y = 0$$

$$\Rightarrow \quad \delta x = \frac{y}{x}\,\delta y \rightarrow \text{virtual displ. of point B.}$$

So, before we go into depth lets derive this condition in a different way. Let's use calculus. So, if this is true, then if I take

$$\delta(x^2+y^2) = 0$$

$$2x\,\delta x + 2y\,\delta y = 0$$

$$\delta x = -\frac{y}{x}\,\delta y$$

So, δy is equal to, so this is the arc length of a circle which is centre at O and radius is OA, and OA is also a constant. So, to a first approximation, this arc length is given by

$$\delta y = a\,\delta\theta \qquad \Rightarrow \delta x = \frac{y}{x}\,a \cdot \delta\theta$$

virtual displacement of point A : $a \cdot \delta\theta$.

So, then we get our virtual displacement of the point A and B. Now, we want to know, we want to calculate that what is the work done. So, what are the forces? So, there is one force P that we are applying to keep this constant. So, the work done, the virtual work done. So, why we are considering the displacement of point B.

First this is movable and second, so, if I think of this slider as a single point particle, and B as its centre of mass, then this displacement of B is the point of application of the force P and we can think of it as the virtual displacement of the point of application of this external force P. So, virtual work done by the force P is

Virtual work done by P: $P\,\delta x = -P \cdot \frac{y}{x}\, a\,\delta\theta$

Now, what are the other work. Other force, so there is no other force in this problem. There is an external torque M and this torque is doing some work. So, one way of thinking about this is to imagine that you are applying some force at point A such that this force is creating this torque. Or you can say directly that because this point, the effect of this torque is to rotate this rod by amount δθ. So, the virtual work done by the other external factor which is a torque is nothing but M δθ. So, this is positive in this direction.

Virtual work done by M: $M\,\delta\theta$

$$\delta U = M\,\delta\theta + P\,\delta x$$

$$\delta U = 0$$

So, then we our total virtual work. Now we apply the condition that this configuration is in equilibrium, hence this variation in the virtual work must be zero.

Now this δθ, from this picture it is clear that θ is an independent variable. So, this is a single degree of freedom system in which only thing that varies is the angular position of this point A represented by θ and then all the other, because of the constraints. The position of all the other parts of the mechanism is automatically fixed. And from this picture y is also the in the equilibrium configuration the y is same as h.

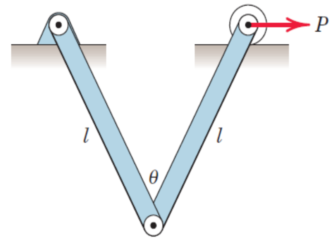
$$\Rightarrow M\cdot\delta\theta - P\frac{y}{x}\,a\,\delta\theta = 0$$

$$\Rightarrow \underbrace{\left(M - \frac{Pya}{x}\right)}_{=0}\delta\theta = 0$$

$$\Rightarrow P = \frac{Mx}{ya} = \frac{Mx}{ha}$$

So, I will give you a practice problem.

Each of the two uniform hinged bars has a mass *m* and a length *l*, and is supported and loaded as shown. For a given force *P* determine the angle θ for equilibrium.



So, this is a pin joint so it is fixed, this is a roller it is movable and if you apply a force P then this roller will move and then this V shape will open up that is the angle θ will increase. If you apply force on the left side then the θ will reduce and this it will close up. So, for a given force P determine the angle θ for equilibrium. So, you first identify, so by applying the principle of virtual work. So, note that this virtual work is not the only way so you can also solve this problem.

Applying your force balance and torque balance condition. It is just an another way of applying, of looking at the equilibrium. And the reason sometimes this is useful because if you do your analysis and free body diagram you have to consider like forces that are the contact forces that are acting here and here and this contact forces the effect of this contact forces is to keep this point fixed. And to ensure that this point the only there is no forces in the horizontal. I mean so this roller moves in a horizontal direction.

And similarly there will be contact force here and those thing force forces will be unknown in general. And the power and beauty of this method is that we do not need to consider those internal forces if we are not interested to calculate them. In this case we are interested in most of this kind of practical situation where this kind of mechanisms are used all you are interested to know is the external force how much load you have to apply on the machine to move it, this is a typical question. So, in this case this is a method which sort of allows you to ignore the internal forces, the whole whose whole effect is to keep the motion constraint. So, let me summarize the method of analysis of solving problems by the virtual work.

So, I hope this systematic summary will enable you to keep to attack any problem where you want to use principle of virtual work. So, the first thing is that you choose your system and now we are thinking slightly differently. So, in your system you identify what are the movable parts and you identify what are the constraints so that you understand the motion of the system. So, the constraint motion of the system. So you guess how the parts move that is consistent with the constraints.

Next part about the interaction so you of course you need to identify all the force and torques on the system and by the surrounding. Now you can and to identify that it helps to draw a free body diagram. So, you should draw a free body diagram as we learned from the previous part of this course. But now you have to do something more. Divide the interactions into two parts the external forces and torques.

So, the forces that can and torques that can do work and in this case the work will be imaginary so virtual work. And the other class force which are constraint forces that do not do any work. And now you think that which of these are known and which of these are unknown. Once Once you have this and it helps to identity to do this analysis it always helps to please draw sketches, please draw some pictures, do not try to blindly apply some formula. Try to draw some rough pictures of your system and in the system in some configurations let us say the equilibrium configuration.

And in this case because drawing picture will help you to identify what are the independently moving parts and it will help you and then you count the degrees of freedom of the system. How many independent coordinates is required to specify one configuration. And to do the further analysis you choose a suitable origin and coordinate system. So, once you have that then you identify the points of application of external forces that do the work.

And then you consider as infinitesimal. Infinitesimal so that you can ignore all the higher second order and higher order term in the displacement. We are going to consider only first order term in the virtual displacement. And these are imaginary displacement about the equilibrium configuration allowed by the constraints. This is very crucial. Then you write down the principle of the virtual work that is the equation and in this case this is we are focusing on work which has a dimension of energy so you have only one equation.

So, compute the virtual work by each external force or it could be torque as we saw in the example. Take the sum and equate to 0 simple. And here one thing that you can check whether your analysis makes sense is that make sure that your equation ultimately should contain the exactly same number of terms independent variables as the degrees of freedom of the system. So, now you take some more example. So with this you with this summary in mind you go and take home, solve the take home exercise and perhaps some more practice problem.

So we shall in the next week we shall take some interesting application of the concept of degrees of freedom and we will consider another third way of looking at the problem of analyzing the equilibrium configuration and we will talk about friction. So, thank you and see you next week.