Newtonian Mechanics With Examples

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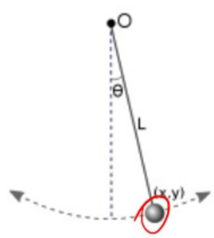
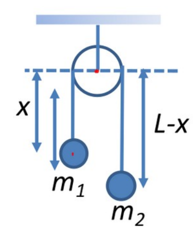
Week -04
Lecture -19

So, this is week 4 of this course on Newtonian mechanics with examples. In the last lecture, we started discussing a new topic, which is the principle of virtual work. In the last lecture, we took several examples of constraint motion from real life, from textbooks, and also from some engineering that may be relevant for different engineering disciplines and also for those of you who will be doing some project or have a hobby in robotics, modeling robots. So, today, we are going to look at how to, what are the different types of constraints. So, the first type we are going to discuss is the constraints that can be written as an equation. Now, what kind of equation? An equation that involves the position that is the configuration of your system, that is, the coordinates, the coordinates that describe the position of different parts of your system and time may or may not be present.

So this kind of constraint is known as by the Greek word holonomic. So, to give you an example, so for example, if you are analyzing the motion of a pendulum, so, in this case, let us say this bob is your system, then this follows a constraint that the length of the pendulum is L is constant. So, that puts a constraint on the possible position of this bob, the x and y coordinates of this bob.

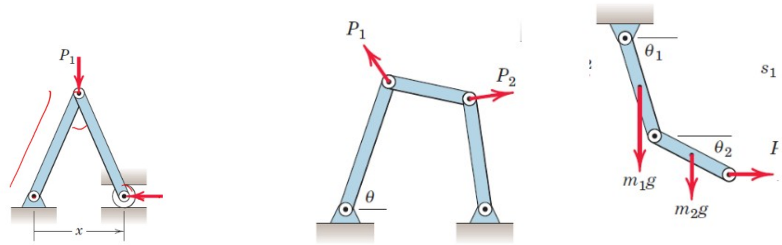




Another example that we also discussed in the last lecture is that when you are talking about the pulley problem, then you have this constraint that the length of the row is constant, which means if the mass has a direction, as a coordinate x, where x is the vertical direction, and we are taking with respect to the distance of the centre of the mass, to the distance of the centre of the pulley, that is x for one mass, and then the other one must be L-x.

So, their motions are not independent. So, together, they move in such a way that the length of the rope remains constant. Another very-very important example is the rigid body, which are multiple connected rigid body. So, let us say this is rigid, so each of these rods is a rigid body in the sense that any two points, the distance between any two points in the rod, we are going to assume, is going to be constant always, no matter what. So, this is true.

Rigid body



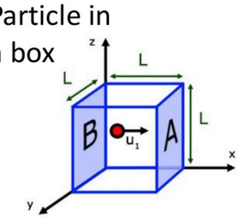
So, in practice, there is no, truly speaking there is no, perfectly rigid body in real life. It is not possible. But we know that there are examples which are approximately very rigid to a very good approximation. The length between the two is very hard. You have to exert a lot of force to change the length of the rigid body.

So, this rigid body, we are going to assume that for simplicity that, the length remains always constant. Then we have several different examples kind of mechanisms of multiply connected rods, connected by hinges. Now, in this case, you see that if I sort of put a force on this roller, this is an example where this one end, so there are two rods connected by a hinge here and the other end of this rod, this left rod, is connected as a fixed point, and another end of the second rod is connected by a roller. So, then you can immediately see that if I exert a force on the roller, then this side, the right-hand rod will start to move, and hence, the left-hand rod will also start to move because they are connected at the hinge. Similarly, if I put a force P1 downwards if you push the hinge downwards, then the angle between these two rods will open up, and hence the rod, both the rod, the right-hand rods the other end of the right-hand rod will immediately start to move.

And I have also shown some more complicated mechanisms with involving more, more number of rods. Then another possible example is a particle which is moving along a given constant path such as a straight line or circle, or it could be even more complicated paths like parabola, hyperbola and sine curve, whatnot. So, in those kind of situations, the equation of the curve is a constraint on the coordinates. So this, the curve that this part bob makes on the, in the space is actually one such example, where the bob is constrained to move along the circular path. Now, but this is not always possible to describe the constraints by an equation that involves the position coordinates.

So, those kinds of constraints that are not holonomic are called non-holonomic.

Particle in a box



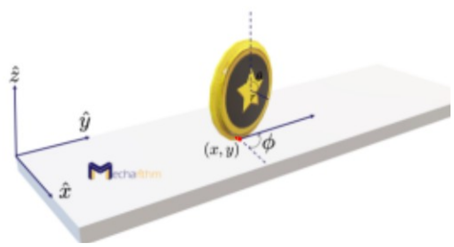
0 <= x,y,z <= L

Inequality, not equation

So, consider a particle in a box. So, this problem you encountered both in various other courses as well. For example, you may have studied this problem in a quantum mechanics course which is a very famous toy model of a bound system. You may have learned or studied this example in your thermodynamics or kinetic theory or ideal gas or the statistical mechanics course where you talk about the ideal gas, classical ideal gas, and these particles are confined in a box.

So, in this case, the constraint is the position; let us say if I take one corner of the box as my origin and the coordinate system as shown in this figure, then the position of the x coordinate can be only between 0 and L, where L represents the length of the box in each direction. But this is an inequality. This is not an equation.

Another example which is also very common and very important is the motion of a wheel. So, such as in this case, consider this vertical disc, which is rolling on a horizontal plane.

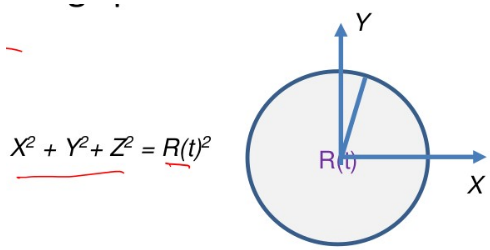


So, the rolling motion. In this case, as you know that, the rolling motion does put some constraints. So, the constraint is that the point of contact, which is denoted as this red dot here, the point of contact must, the relative motion between the instantaneous point of contact and the surface on which it is rolling must be 0. But this is a condition on the velocities, not on the position of the, does not matter where the wheel is. It is a condition on the velocity of the point of contact. So, this is a constraint that constrains the velocities of particles but not their position.

And if we further analyze mathematically, we shall see that there is a difference between the equations that involve the velocity versus the equation that involves the position. So, this is an example of a non-holonomic constraint. Now, you can also classify the constraints in a, from a different point of view. So, you say that we have a type of constraint which is not an explicit function of time. So the constraint is independent of time. It is not changing with time.

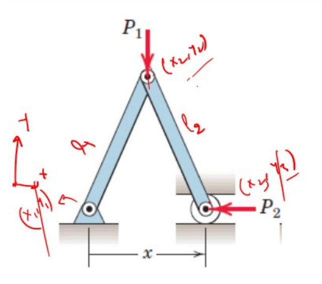
This kind of constraints are called Scleronomic constraints, which means the point is that this way of looking at the constraint is independent of holonomic versus non-holonomic. So, this kind of constraint can be both holonomic as well as non-holonomic. And similarly, the other possibility is that the constraint that is an explicit function of time, and those are called the Rheonomic constraints. Again, they can be both holonomic as well as non-holonomic. So, this is a simple example.

Consider an ant on an expanding sphere. So, if the sphere is expanding, then its radius is a function of time. So, the coordinates of the ant on the sphere satisfy, not all the coordinates are independent. They must satisfy this condition that the sum of the squares is equal to the radius square. Now if this radius is fixed, then this will be considered as a Scleronomic constraint. And if this is an expanding sphere or a contracting sphere, that is a radius is changing with time, then it will be a Rheonomic constraint.



Now we come to the first effect of constraint motions. that what this example shows that if you have a constraint motion, then the coordinates, the positions, the coordinates that describe the position of your system are not independent. Then the, naturally we ask the question that how many coordinates are independent actually. So, for example, if a pendulum problem, if there is constraints like that, so you have two coordinates to describe the position of the bob of the pendulum and they are related by this equation, then only one of them can be independent. So, then you count the number of independent coordinates required to uniquely specify a configuration. So this is called the degree of freedom.

So, for example, here I take this example. So here you may say that you can think of it as a three-point system or a two-rod system. So, if you think of this as a three-point system, then how many variables? So, when I say coordinates, is basically means variables, unknown variables in general required to specify that this is one configuration of the system which is allowed configuration of the system. Now, how do we describe this configuration? Well, you may say that we, let us say this, have some coordinates, and this point has some coordinates and this point has some coordinates. So, by looking at the picture, it is clear that the z coordinate does not play a role, so the motion is effectively a two-dimensional example.



So, I am ignoring the z coordinates. However, so this x_1 and y_1 is a fixed point, it does not change. So these are actually not relevant to describe, so in all the different possible configurations, they remain the same, so they are not relevant for description. Now, this x_2, y_2 and x_3, y_3 they will of course, you need to specify them, but note that they are not independent, why? Because there is a constraint that x_2 and y_2, for example, you can easily see that if this point starts to move downwards, then immediately this point will start to move right side and if its move starts to move upwards immediately, it will start to move left side. And again, similarly, if you put some start, if you change the position of this point, then this point will also automatically change.

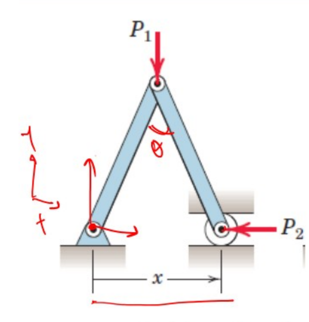
So, why are they not independent? Because this distance between these two points represents the length of this rod, and this is a constant, so this is one constraint in the system. Now in this case, because this rod is allowed to move only along the along this channel, so, that means for all the possible allowed configurations, the y_3 is also a constraint. So, you can drop y_3. So, you have only the possibility that you have this rod x_3 and yeah, so y_3 is a constraint, So you can drop that and this l_2, so l_2 is a constraint. Similarly, there is another constraint, let us call that l_1, which is the length of the other rod.

So, this motion is also determined by the fact that it is connected to this point. This is a fixed point through this another condition that l_1 is equal to constant. So, we see that even though we have three variable coordinates x_2, y_2 that determine the position of this point and the x coordinate that determines the position of this point, there are two constraints between involving those coordinates. So, for example, here we can write-

$$\left.\begin{array}{l} l_2 = \text{constant} \\ l_1 = \text{constant} \end{array}\right] \begin{array}{l} (x_3-x_2)^2 + (y_3-y_2)^2 = l_2^2 \\ (x_2-x_1)^2 + (y_2-y_1)^2 = l_1^2 \end{array}$$

So, this involves two equations, so these coordinates x_3, y_3, sorry x_3 and x_2, y_2 are not independent. So, effectively, there should be only one independent coordinates.

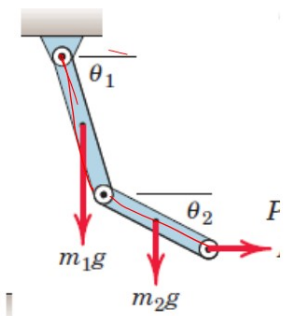
So, this even though it looks a complicated object but, if you analyze them carefully, you will find that this is fairly simple then this is just a one degree of freedom. So, that means only one coordinate, one variable is enough to one to specify the position of this system. Now the point is that but this variable is not a unique choice. For example, in our analysis, we considered the Cartesian coordinates, so, let us take that with respect to some arbitrary origin. So, we take the Cartesian coordinates of let's say this is our y-axis and this is our x axis; draw it here.



So, let us say this is my origin and this is my y-axis, this is my x-axis. Now in this case, so we took the Cartesian coordinates but you can, you do not have to take, that is not the only choice. For example, you can also take the distance between this support and this support as shown in this figure as your independent variable. And you can immediately see that if you know this, specify this distance then which is let's say if I take this as my origin then this is nothing but the x_3, then because of the other two equations, the position of this point is automatically fixed. You can also take, but you do not have to take that distance also, you can also take this angle as your variable with this angle plus the distance of this rod you can immediately see that this x is automatically fixed.

So, you can also denote this angle as your independent variable. So, this sort of generalizes your Cartesian, you do not have to restrict it to the Cartesian coordinates. You can take any convenient coordinates, coordinate variables to represent the configuration, and describe the configuration of your system. Anything is okay as long as it is unique and it has the right degrees of freedom. So, such kind of generalization of your basic Newton's, Newtonian way of description is called a generalized coordinates to describe the configuration.

So why generalized? Because note that if I describe this condition using this variable x, it has a dimension of length, its position. Whereas this angle theta is an angle, it is dimensionless. So your nature of your coordinates can be anything as long as they describe the configuration uniquely. Now this is another slightly more complicated example in which you can, so I sort of leave it to you as an exercise to sort of analyze, and identify what are the constraints in this case-

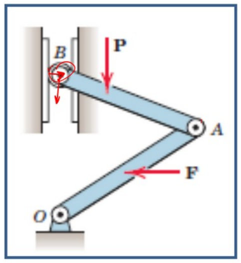


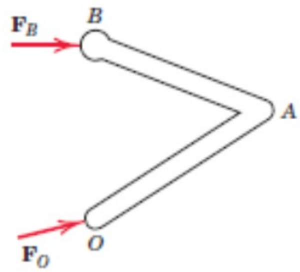
So, one constraint will be the, that the length of this rod is fixed and length of this fixed and convince yourself that in this case in order to describe the position of this arm, robotic arm uniquely, you need two degrees of freedom. That means there are two independent variables required to specify the configuration. Hence it has two degrees of freedom and one choice will be the angle that this rod makes with the horizontal direction and the angle that this rod makes the horizontal direction.

Now the second effect of constraints is what led to our principle of virtual work is that the constraint forces are unknown and we are sort of in this kind of situations we are dealing with, we want to get rid of them. We are not interested to know the, to calculate the unknown forces. So, this is a scenario where we are not interested in the unknown constraint force.

Note that in the previous example of truss, we were interested to know the contact forces on the joint because that is what we want to know that how much force the joint can support. In this, but now we are dealing with certain situations where we are not interested to know the unknown contact forces, and we want to get rid of them. So, these forces are self-adjusting forces, and they are not known beforehand. So, the key insight here is that these kinds of internal self-adjusting forces, the constraint forces, do no work. So, let us analyze this statement carefully.

So, consider this example of again this mechanism of a hinge. There are three kinds of constraint forces you can identify here. So, let us take them one by one. There is a constraint force between this roller. So, what is this mechanism? So, this roller can move up and down in this channel, and these two rods are connected at a hinge A, and then the other part of the lower rod is connected to a fixed point O.





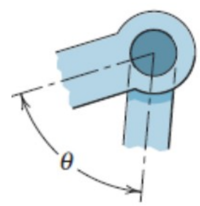
So, this is the mechanism. This is a half-scissor-like mechanism. So, if you move this roller up and down, then this point A will move, and the angle will change. So, the point A will move up and down, and the angle will change. So, this is the motion. Now the constraint force, the contact, so what are the external, if you take this mechanism OAB as your system, then there is an interaction between, so let's take so there is a contact force, contact interaction at point B between the wall of the channel and the rod.

Now the effect of this constraint is to ensure that the roller moves only in a vertical direction, which means that there is no horizontal force, sorry, so no horizontal displacement possible for the roller. That is the constraint. So that means that this contact force, so, let us assume that there is no friction between this channel at this surface. So, then the contact force must be in a direction which is normal to the surface and the displacement of this roller is vertical, so they are in the perpendicular direction. Hence F_B does no work, because the displacement is perpendicular to the force.

So as indicated in this figure. Now, there is another force at this joint O which is fixed. So, then, as you know that in this case the force need not be in the vertical direction, it can in general has a component in the vertical direction as well as in this direction let us call that x direction and y direction. However, this constraint force ensures that the spring joint O does not move. Again the, whatever contact interaction between our system with this spring joint at this point O does no work because the displacement is 0. Now the third constraint force acting at the third interaction between our system which is OAB with our surroundings which is this hinge A is the contact force at the hinge A.

So this one may be slightly difficult to visualize, but there is a contact force pair that acts on the one between the two rods and this is kind of maybe, say, an internal force. So, this one force, because of the way they are moving, this force or direction is shown in this figure and so they are by Newton's third law their action-reaction pair and they the constraint is such that these they must have identical displacement in the because this is point A. If point A moves, then it is the same displacement for both cases and by Newton's third law, these are action-reaction pairs so they are equal and opposite. That means that the work done by one force must be equal and opposite because the displacement of the same. So the work done by one this force and this force must be equal and opposite.

So their total sum is again 0. In fact, this force itself F_A can be 0 in some situations. Now I must point out one thing is that in practical situations, friction is always present. So we are going to discuss how to think about friction perhaps in the next week, but we all know by experience what friction is and we are going to consider two kinds of friction the sliding friction, which is always present at joints and hinges. So this is by which So, this is by which the two if you have two surfaces they are trying to slide related to each other.



So this is the sliding friction. So, if I take this pen and this is a surface, then this is a sliding motion. Now, this force does some work. The work done by this sliding friction force is not 0 in general. So, this is in fact a counter example the presence of such sliding friction that internal or the constraint forces does not do any work. So the principle of virtual work that we are going to consider next which is actually a statement about the energy conservation so will fail.

So, this is a disclaimer or warning in the form we shall consider in this course. If dissipative forces, that is like sliding friction is present, we can correct it by adding the work done by the sliding friction in our equation, but for simplicity, we are going to ignore the sliding friction here. So, we are going to state the principle of virtual work, and in the example that we shall work out, we will assume that there is no sliding friction. Now, the rolling friction is okay.

So if this is the pen so this is a rolling motion. So, the rolling friction without any sliding is okay because this is a case where the points by definition for in the case of pure rolling without sliding, the points of contact are at rest momentarily. So it can do no work, which is consistent with the rolling friction. So, we are going to ignore sliding friction. Rolling friction is probably is okay. So, now we are ready to consider to state the principle of virtual work.

So, this is going to be the topic of the next lecture. Thank you.