Course Name: Newtonian Mechanics With Examples
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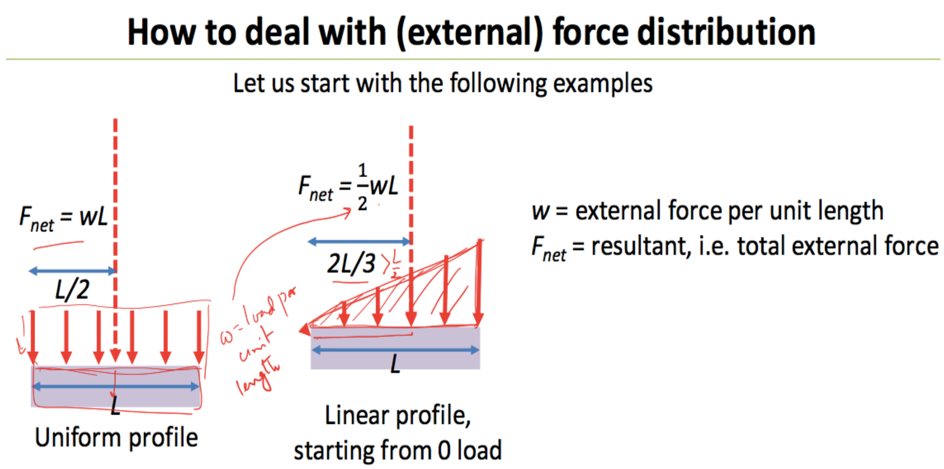
Week 04
Lecture - 17

Let us continue our discussion on distribution forces that we started in the previous lecture. So just to remind you, the question is: suppose you have an element of a truss or a beam such as what is shown here and we want to know what are the forces acting on the support, and In this case, the external force is distributed over the entire length of the force. So, we want to know, and we want to replace that with a single force acting at some particular location and the question is how to do that. So let us start with the following examples: So here is a case of the uniform beam. So these little arrows represent the load per unit length. Now given this length, the load that is distributed over a length L, what is the equivalent force? So what is this equivalent force mean? It simply means that if you want to replace this uniform profile of load with a distributed load, then by a single load, a single equivalent force, they must have, and the total force must be the same.



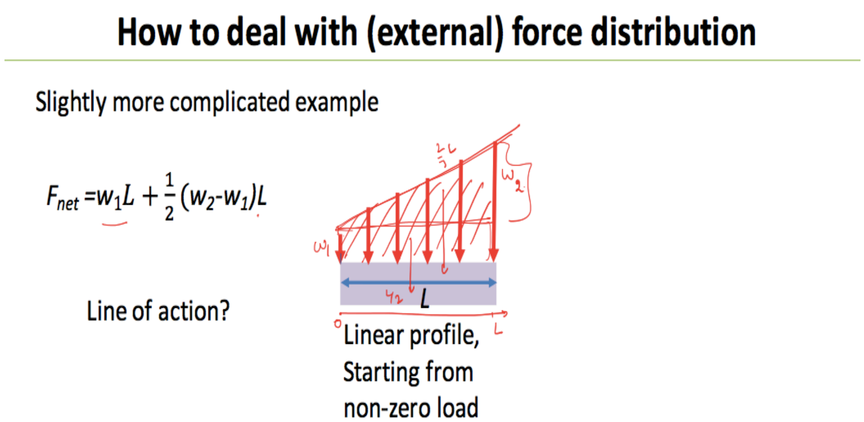
This is the first general principle. In order to do that, of course, in this particular situation, it is very easy to calculate the total force. So if these little arrows represents load per unit length and they are in this 2D example, Let us say 1D, one-dimensional example, so this is distributed over a length L. The total force must be the area of a rectangle whose one side is L and the other side is W, which is the load per unit length.

So then the total force must be W times L. So remember this W per unit length: So it is force per unit length times length, so dimensionally, it makes sense that W times L has a dimension of force. Which represents the total force. Now the question is, where is this force you should put this force? Now, in order to determine that, the general principle is that the total moment generator, The total torque produced by this force at a given pivot point that you have selected must also be the same. For the case of this distributed force and the equivalent force, Now, in this case, it is easy to see that just by symmetry, because everything is uniform, Then you should put this force precisely in the middle of the beam, just by symmetry.

In this case, let us now take the second example, where you have a linear profile starting with a zero load. So in this case, again we can determine the total force easily from geometry. So, again, we can use the same principle that the total force must be, so if you have W per unit load, But now the Ws are different, but you can still see that the total force must be the area under this curve. So this area under this curve is a triangle, so this represents a length L and This is force per unit length, so this area represents the total force. And this area is easy to calculate; this is the area of a triangle, so this is half times the base times the height.

So, the total force is half WL, where W is the load per unit length. However, in this case, it is not so obvious to see where the line of action is and where you go with the net force. Now we can obviously have a slightly more complicated example where we can sort of have a linear profile. But starting from a non-zero load. Now in this case, again, if you want to compute this total force, we can sort of use this geometrical strategy that this total force must be the area because these arrows are load per unit length times the total length, which will give you the total force.

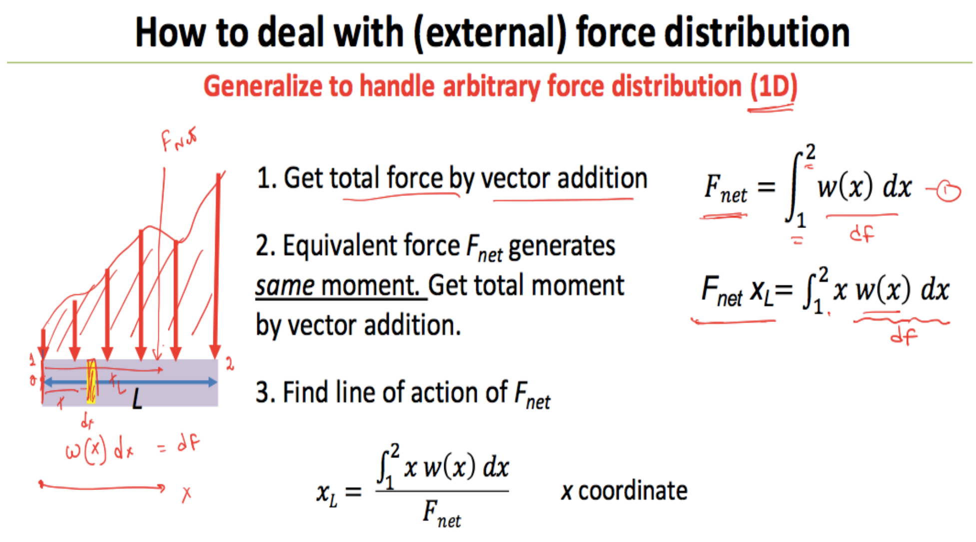
And in order to calculate that, you can simply divide, so it is kind of a combination of the previous two examples. So, the area must be the area of this rectangle. So let us say this is at the one end, let us call that, let us say this is my x axis and this is my origin and this is L, so at x equal to 0, the load per unit length is W1 and at x equal to L, the load per unit length is W2, then this area is under this curve. This straight line is the area of this rectangle plus the area of this triangle. So area of the rectangle is W1 times L and the area of the triangle is given by dividing the height by half.



Which is W2 minus W1, so this is W2 minus W1 times the base, which is L. Now again, but where is the line of action? So, this is somewhat, maybe not easy to show. So in the previous case, I have shown you the answer, so the line of action is passing through a point. Which is not at the middle point, because it is not at the middle point. Because the forces are not uniform, you can see that, so this is at two-third distance from one end, and why? So, this is slightly bigger than the middle, because does it make sense? So this is again: I must mention that whenever you are analysing a mechanics problem, One question that you should keep asking yourself is: Does it make sense? So in this case, it makes sense because the loads, as you go further from the left end, The load per unit length increases, so that means you expect that there are more end towards the right end.

So, the line of action of the force should be biased towards the right end. Hence, you see that the line of action is bigger than the middle point. Because of this, with this example, now in this case, the line of action is not so easy Because if you look at only the rectangle, you expect the line of action to passing through the middle point. Whereas if you look at this triangle, you expect the line of action passing through the two-third distance. So this is L by 2 or half L, this is two-third of L and then if you combine them, It is not easy to see how to get to the line of action.

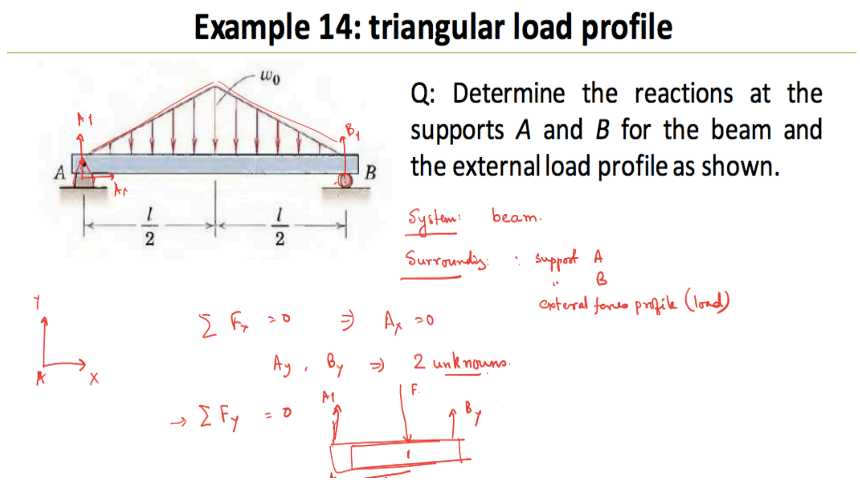
So then what we want to know is how to handle any arbitrary force distribution or distributed profile in one dimension. So this is a one-dimensional problem such as this, so this is a completely arbitrary function, some non-linear. So in those cases, what is the general physical principle? And the physical principle is the following: So first, to get the total force, that is the equivalent force, which is the total force acting on the beam, So, this must be the equivalent force; it must be the total force. So this is what you get, so if you consider a small element such as this highlighted region, the yellow region is dx. Then, as I mentioned before, if you know this load per unit, load profile, Load external force per unit length, this is W, then W times dx is now your force acting on this particular earlier element.



Then the total force is actually the area under this curve. And hence, so if you start from, this is your point 1, this is your point 2, So then your total force must be, which you can compute simply by adding vector addition. In this case, all the forces are in the same direction, so this vector addition turns out to be a simple scalar addition of the magnitude. So this is dF, and if you add dF for each little piece, then you get the total force. So this is it, and I put the two ends, 1 and 2, to remind us that this beam is of finite length and you go from one end to the other.

The second principle is that the total moment is distributed, so if you choose any particular pivot point and If you calculate the total moment or total torque generated by this, this profile of external force, distributed force, they must be equal to, So that to the moment generated by the equivalent force, point force, that is F net. So what is the moment generated? Suppose we want to compute the torque above any point on the left hand, left end, So, let us call that origin, and this is our x-axis. So this is our pivot point at the left end. Then suppose the net is the total force, or Fnet is acting at some point xL from the left end. Then the moment, as you can see, that we have generated by this force F net is about this point origin, pivot point O must be Fnet times xL in one dimension because This magnitude is going to be; you can find the magnitude from the right-hand thumb rule; it is going inside the screen.

And what is the total moment generated by this distributed force? So, this much is the force acting; if you take an element which is at a distance x, then at a distance x, if the load per unit length is W, which is in general some function of x, then W times dx is the dF, and then the distance is x. So the x times dF that gives you the torque generated by this particular piece and Then you can get and as you can see, The torque generated by each of the different pieces at different parts of the beam all points in the same direction. So you can simply add them in an algebraic way and then you get the total torque and because this is a continuous function, so instead of summation, we are going to use integration. So this is the strategy. So now, once you know the Fnet from this first equation and then you can use the second equation to compute the distance or the position of the line of action.

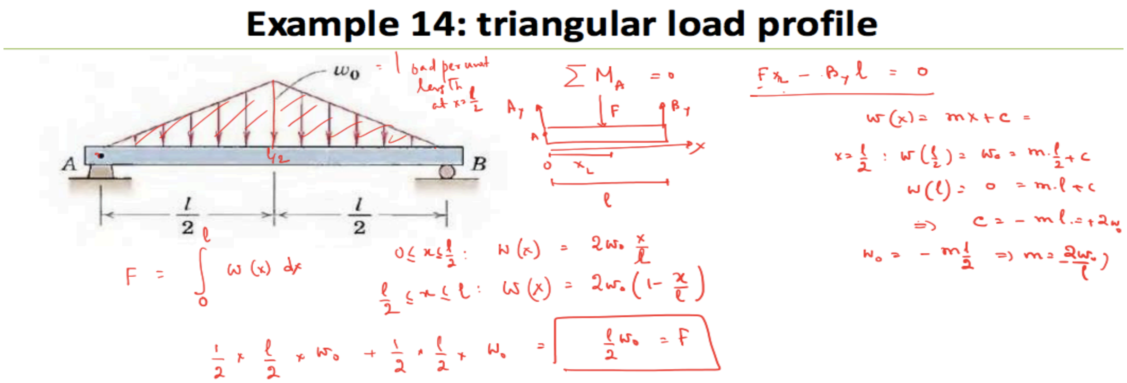


So let us take an example. Work out an example to understand. So here is a question: there is a beam given which is supported at both ends. On the left-hand end, this is a pin joint, which means it is kind of fixed, and on the right-hand end, you have a roller. So that means we are assuming so these two supports has different shapes, different types of support.

In this case, there is no horizontal force; the only force must be in the vertical direction. So, determine the reactions at supports A and B for the beam and the external, and this line represents the external load profile. So this is given, and this design is given. So the question asks what the forces are that support A and B. So let's first analyze this in detail.

So the first thing is that what are the direction of the forces. So first, let us start with a coordinate system. So let us say that we take point A as the origin of our coordinate system and this is x axis and this is y axis. Now, the nature of the support determines that, in general in this particular case the force that are acting at this point has two components. Let us call that Ax, and let us call that Ay.

Whereas at the end B the only possible component is By and the external force are given. Now, the first is that you do not have to remember a very complicated formula. You just have to remember the simple physical principles. So let's first write the force balance in the x direction. So the whole structure is in mechanical equilibrium, so this equation represents the force balance in the x direction.

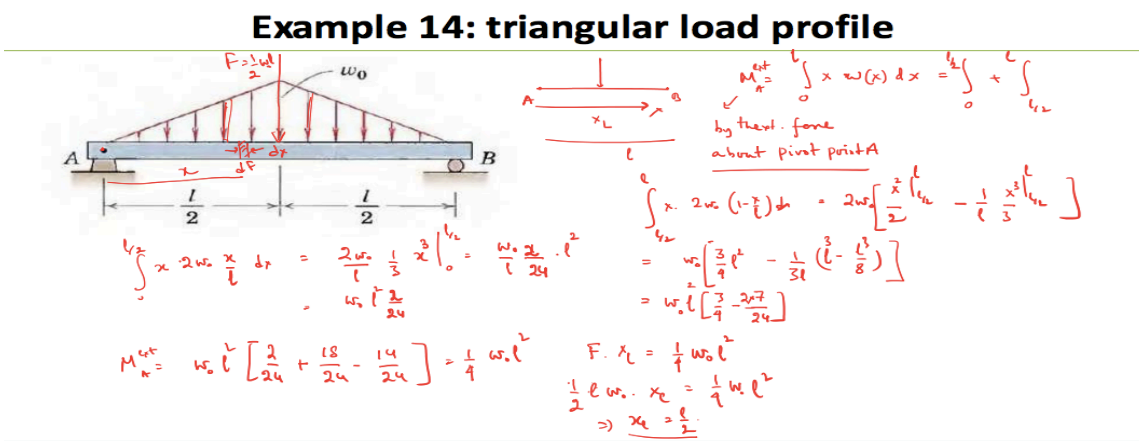


Now you can see that only horizontal force is acting, okay? So let us do one more thing before that. So what is our system? So let us say our system is the beam that is the red block. Then what are the surroundings? So the important thing in the surrounding is support A, and support B and the external forces profile load. Now note that in this case, since nothing is mentioned, we are going to assume that the beam is massless. So, we are going to ignore the interaction between the beam and the earth.

So now we write down the force-balance equation. So the first equation is in the x direction. Now, the only possible force in this case is that all the external loads are in the y direction, so no component in the x direction. So this means that the only possible horizontal force is Ax and Since the whole thing has to be motionless, it is static, which means the Ax must be 0. So that means we have only two unknowns: one is Ay and the other is By, so these are the two unknowns.

Then we can write the force balance equation in the y direction and then what we get if we write down the force balance equation in the y direction? We get that, so what are the forces in the y direction acting on our system? So one force is Ay, other force is By and then there is this distributed force and in our free body diagram, So let us say this is my beam, so then we have one end which is Ay other end which is By. Then we are going to sort of assume we are going to replace this entire distributed force by a single external force F and this is our origin and Let us assume that the line of action of the force is situated at a distance xL from the left end, or support A. So this is the situation. So now this is kind of one equation, so we have two unknowns, so we need one more equation and this third equation is going to be the sum over M A, so M represents the total torque acting on our system and I have put this suffix A to remind us that this torque is calculated around this pivot point A, so this must be 0. Now if this is 0, we draw the picture, so this is Ay, this is By and this is some F This is the equivalent the total force acting on the total external force acting on the beam and This is at distance xL.

Now if we want to calculate the moment, So about this point A, Ay does not contribute because its line of action is passing through the points A, F, and By will contribute because their line of actions are not passing through A. So, then the total torque given by F is F times x L because this is from the by definition and The torque given by By is By times L, so this distance is L. And as you can see, if you apply right-hand thumb rule, they are in the opposite direction. So they are cancelling each other and hence the total torque is 0. So that is required to ensure that the beam does not rotate.



So now you have two equations, two unknowns and Now we can compute these By and Bs. Now note that this F and xL are actually known Because this external force is given, the unknowns are By and Ay. So how do I get F? Now, if I apply the definition that we just discussed, then F is given by and lets us go from 0 to L, so this is one end, which is 0, x equal to 0. Let me remind you that this is my 0 and this is my x axis. Now there are two ways to do that, first way is to sort of easy way and the lengthy way, So anyway, the first thing is that we want to know the Wx, so how do I get the Wx? So this is starting from 0 and this is a linear, clearly a linear profile, It is better to sort of divide it into two pieces, one piece from 0 to L/2, the midpoint, and the other piece from L/2 to L.

So this is clearly an equation of the form y equal to mx. So you can see that the slope of this equation is from 0 to L/2. This Wx is given by; now the slope is given by W naught divided by L/2. So that is 2 W naught by L, so W naught is the load per unit length at x equal to L/2. So, does it make sense? that if you put x equal to 0, you get 0, and if you put x equal to L/2, you get W naught, so this is fine.

Now for the other case for the second piece, that is from L/2 to x less than L, So we can find out by sort of assuming that this Wx is a straight line of the form mx+c with two unknowns. Then we have two conditions: at x equals L/2, then W is naught, which means m times L/2+c is equal to W naught and W at L, the right-hand end is 0, and the load per unit length is 0. That means m times L+c is 0, so this gives me c, which is minus mL and Then, if you plug it back in this equation, first equation then you get W naught is equal to minus m L/2 Because m L/2 minus m L is minus m L/2, That means m is 2 W naught by L with a negative sign, Does it make sense? In this part of the slope, you can clearly see that the slope is negative, so we can check that the right-hand side is negative. Then our equation is now: if we use this condition here, then we get c, which we can write out, so m times L Which is minus 2 omega naught plus 2 omega naught, so then our profile is 2 omega naught 1-x/L and We can verify that at x equal to L, this is 0, and at x equal to L/2, this is W naught, so this is our W x. So now we can go a lengthy way and you can plug in these two equations and do the integration or we can look at geometrically the quicker way, as we can see that the total force is the area under this curve and The area at this curve is nothing but the sum over area of this triangle and the sum over area of this triangle.

So the triangle on the left has an area which is half times the base, which is L/2 times the height, which is W naught. triangle on the right has an area half into base; again, the length is L/2 into height W naught. So I get half into L/2 times W naught. so this must be the total force. Now, next task is to compute the total torque at generated by this external profile.

So this total torque is our one important equation, so next task is to calculate the total torque. So let me again write down this picture this is A, this is my beam. So this is the x axis; this is A; this force is acting at a distance x L; this is B; so this length is L. So the total torque, as we can see, is x time dF, so if this distance is x, This distance is x, then this piece about this point A, so let us say this force acting on this dx part of So this is a length dx, a small slice dx, and then the total force is dF and The dF generates torque, which is x times dF, which is W x times dx. Now this integral goes from x equal to 0 to x equal to L, so this is the total torque by the external force about the pivot point A.

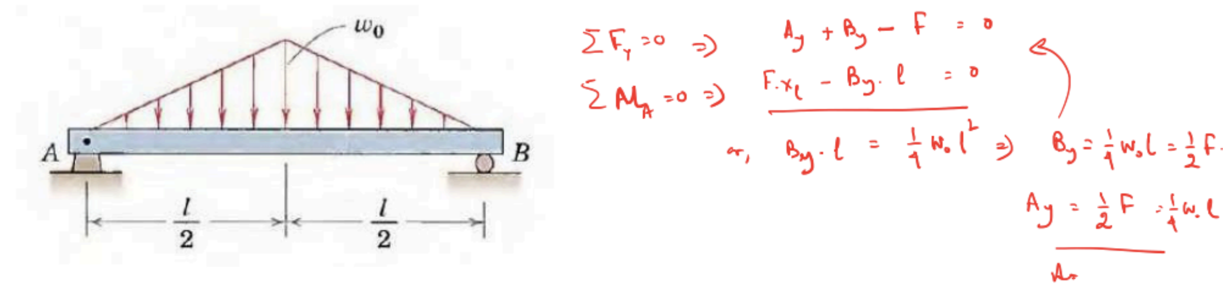
So when you sort of write these answers, to these kind of questions, You should always write these details; It is very important to write these details because they will help you make sure that your calculations is correct. It also helps you remember the basic physical principles. So there is no way we have to do this integral, so for this integral again, We divide into Wx as two pieces, so we divide into two part, so the first part, So, 2 W0 x by L dx, so this is equal to we can take 2 W0 by L outside, we have x square dx, So integration will give us x cube by 3 and 0 to L by 2, so that will give us L cube by 8 times 3. So 1/24, so that will give us W0 by L times and so this L and there will be L cubes. So we will get an L square 1 by 2 by 24, so this 2 comes from top and 1/3 divided by 1 by 8 is 24 times L square.

So that will give us W0 squared by 2 by 24. Now, the other term and in this case the first term, has x dx which will be x square by 2, So this is x squared by 2 and L by 2 to L is the limit of the integration. The second term has a sort of 1 by L sitting outside and then you have x squared dx, so which gives us x cube by 3 L by 2 to L, so this will give us, so this 2 will cancel and then You have L square minus L square by 4, which is 3 by 4 L square, and we have minus 1 by 3 L times L cube minus L cube by 8, which is 7 by 8 times L cube and The second term is L, which I will cancel and get an L square, which I am taking out. So this is 3 by 4, minus 7 by 24. So if I add these two pieces, We get the total M as W0 L square 2 by 24 plus 3 by 4 is 18 by 24 minus, so this is not 7 because there was a factor of 2. This 2 was inside, so this must be 2 into 7, because of this 2 being inside.

So this must be 14 by 24, and if you simplify this, you will get 1 by 4, so the total torque is not L square. So then we can sort of look at our equation for torque balance. So we can now calculate this: this is our external total torque, MA and now we can sort of determine what xL is because F times x L is equal to this, Which must be the total torque acting on the system, which is 1 by 4 W0

L square. Now, F is what we just computed here F is 1 by 4 W0 times L, so this is half 1 by 4, so again, I made a mistake here. So in the previous slide, it should be 1 by 2, because there are two factors of 1 by 4.

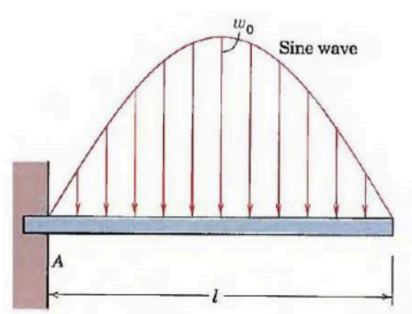
Example 14: triangular load profile



So, if we add them, you get a half; half L times W0 times xL is equal to 1 by 4 W0 L square, so from here you get x L to be half L. So, does it make sense? so you determine that you can replace this external force with a magnitude of total force, Which is half times W0 times L, and this is acting at a distance precisely at the middle point. Does it make sense? well the line of action do make sense because by symmetry, you expect that if you take any force, the line will be about the middle point. Any part of the force distribution is sort of symmetric about the middle point. So by symmetry, we expect that the line of action of the force should go through the middle point, But you have to do this calculation to actually verify that.

So now we can determine the unknown forces B, Y, and L because we have calculated F and x L. So what we have is two equations, Our first equation shows that, from here, our first equation, Let me write down the first equation again. So from this equation, We got that Ay plus By and minus F is 0 and from the second equation, the torque balance equation, We get F times x L minus By times L is equal to 0, so the sum over torque is 0. Now, from second equation, we have determined that F times x L is given by 1 by 4 W0 L square, So By times L is equal to 1 by 4 W0 L square, then we get By is equal to 1 by 4 W0 L, which is half of F. Now if you plug in the force balance equation, you get that Ay is equal to half of F as well.

Take home exercise: sinusoidal load profile



Q: Determine the force and moment reactions at the support *A* of the built in beam subjected to a sine-wave load distribution as shown.

So now there is a take-home exercise, so there is a similar problem, Determine the So given again a beam, now this is supported at only one end, so this is a cantilever. So this is given, and the external force distribution is now a sine profile. Now, given this situation, then you determine the force and the torque that is acting on the support A. So I will leave it to you as a take-home exercise. We will continue discussing more problems in the next lecture. Thank you.