Newtonian Mechanics With Examples

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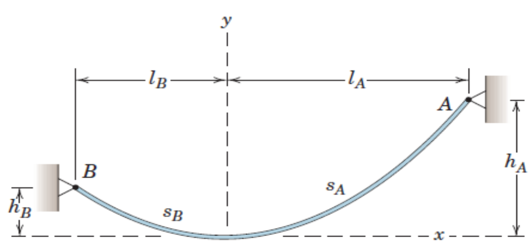
Week -03

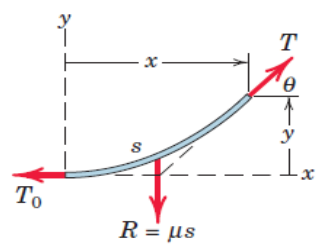
Lecture -13

So, in lecture week 3 of this course on Newtonian mechanics with examples, we were discussing this problem of the suspension cable. So, in the previous lecture, we took a cable which are supported by both ends. And in that case, we assume that there is some external vertical load acting on the cable, but the cable itself is massless. Today, we are taking another example in which we are going to assume that there is no external force apart from, but the cable is now massive. So, it has its weight.

We cannot ignore the weight. So, the cable, which is flexible, is hanging under its own weight. So, let us define the problem.

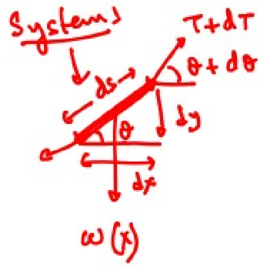
A flexible, uniform cable of length *s* is suspended from two ends *A* and *B*. It is hanging under the action of its own weight only. μ is the mass per unit length of the cable. If the structure is in equilibrium, then what is the shape *y(x)* of the cable?





So, note the difference in the previous example that we took. In that case, the external load was a w_0 per unit horizontal length. In this case, the external load, the mass per unit length, is per unit length of the cable. So, there is a crucial difference. Now, again, the structure is in equilibrium. Then what is the shape of this cable? Now, in the previous case, we found in the case of massless rope, which is hanging because of some constant external load per unit length, the shape was a parabola.

Let us see what we find in this case. So, let us recall that from the force, from our analysis of by applying the condition of force balance, what we found is that suppose this is my system, and let us kind of take a small piece. So, we took a small piece of the rope, and this small piece of the rope at this end is kind of changing. So, the tangent direction makes an angle $\theta+d\theta$ with the horizontal, and θ at one end and $\theta+d\theta$ at the other end. The magnitude of the tension force is changing in this case because of the vertical forces.



And now we have s/ome vertical force, which is the weight. So, some vertical force, I am going to write it down. So, let us call that w and this is the horizontal, so this is the length of the rope. Let us call that ds, and this ds has two components, dx and dy. So, now what we found from horizontal force balance, we found our first equation that the change in the horizontal component of the tension, as we go from this end to the other end is 0. So, the horizontal component was constant. So, this came from the horizontal force balance.

$$d(T\cos\theta) = 0$$

And from the vertical force balance, we got this constant that the change in the vertical. The tension on the vertical direction was balanced by this external force. Now, in general, this w can be different, at different point of the rope. In the previous example, we considered that w was constant. However, in this example, this w is the weight of the length of this slice of the rope. This is our system, which was the slice of the rope. Now this w(x) dx, so that was the what we found. Now, in this case, what is the weight of the slice of the rope? So, the per unit, so the mass is, the per unit length is μ, and the length of the rope is ds; of our system is ds, so this is the mass. If you multiply by the gravitational force, that is what will give us dw, the weight of this mass.

$$\underline{d(T\sin\theta)} = \omega(x)\,\underline{dx}$$

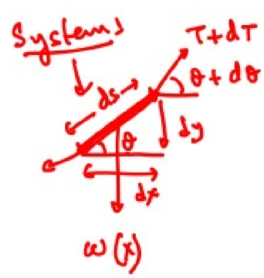
$$g(dm) = g\mu\,\underline{\underline{ds}}$$

Now note that this ds is not same as dx. In fact, from geometry, we have,

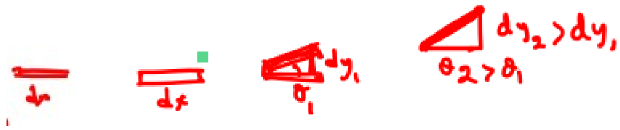
$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\text{weight of our system } \left[\mu g\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right]\cdot dx$$

The curvature is changing at different point, continuously changing, so $(dy/dx)^2$ is not a constant. This represents the slope of this rope, which you can see from this picture, continuously changing, which means our effective load per unit horizontal length in this example is no longer a constant, but it is changing with x.



How can we understand this? You can simply think of it this way that: if I have the same amount of dx, the horizontal component, but at a different; consider two pieces of rope, which has the same amount of dx, but one is the angle makes with the horizontal is θ_1 and the other one is a little bit more bent towards the upward direction. So, this is θ_2, θ_2 is greater than θ_1.



Now, if they have the same amount of dx, but the vertical elevation of this and this piece are different. So, the vertical elevation is more, which means to cover same amount of dx, rope length of this piece with a slightly higher elevation must be longer, which means this mass must be longer, and hence its weight is also greater.

And this is reflected by this factor, the slope is $(1 + (dy/dx)^2)$. So, this is the effect of the mass of the rope. So, then, in the previous example, we got this equation. So, we had a third condition. So, we wanted to eliminate θ and replace θ by the y. So, we had this third condition that

$$\frac{dy}{dx} = \tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{T_y}{T_x} = \frac{T_y}{T_0}$$

And if we take the second derivative, then we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{T_y}{T_0}\right) = \frac{1}{T_0}\frac{dT_y}{dx} = \frac{w(x)}{T_0}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\mu g}{T_0}\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

So, our equation for curvature, which is a second derivative of the y as a function of x, in the presence of a, if the rope is massive, is now slightly different. It looks more complicated than the case where the rope is massless. So, and this effectively comes because the load, the vertical direction, the weight is the load which is not constant per unit horizontal length, but this is constant; I mean if the rope is uniform, then the mass density μ is constant. So, it is constant per unit length of the rope. So, this is what makes the difference.

So, again, to solve this equation, we need the boundary condition and the boundary condition as before. We will set at origin, we will choose the origin at the middle, the lowest point of the rope,

Boundary condition:

at $x = 0$ $\quad y = 0$

$\frac{dy}{dx} = 0$

Now if we sort of how to solve this equation, so let us assume that, make a change of variable that dy/dx is z, then I can write it as a first-order differential equation,

$$z = \frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{\mu g}{T_0}\sqrt{1+z^2}$$

$$\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\mu g}{T_0}\, dx$$

$$\sinh^{-1} z = \frac{\mu g}{T_0} x + C$$

$$z = \sinh\left(\frac{\mu g}{T_0} x + c'\right)$$

$$\Rightarrow c' = 0$$

$$\frac{dy}{dx} = \sinh\left(\frac{\mu g}{T_0} x\right)$$

Now, what is a sin hyperbolic curve? This is a quick recall here,

$$e^{x}, \quad e^{-x}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx}\cosh x = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

So this looks similar to your famous trigonometric identity that $\cos^2\theta + \sin^2\theta = 1$ with the difference that this is now a minus instead of plus. So, this is not a trigonometric function cos or sin; this is a hyperbolic function, hyperbolic cos and hyperbolic sin. So, this is a different function, so this is the definition for your reference. So, we are not going to use so much about, so nowadays there are a lot of standards plotting tools available, so you can after this lecture you can go back and use your favourite plotting software and plot this function and see how it looks like. So, one crucial difference for example with cos and sin function is that cos and sin function is a periodic function with some period 2π, its values get repeated after 2π, whereas this is not a periodic function. Now, if we integrate this further,

$$z = \frac{dy}{dx} = \sinh\left(\frac{\mu g}{T_0} x\right)$$

$$y = \int \sinh\left(\frac{\mu g}{T_0} x\right) dx + C_1$$

$$= \frac{1}{\frac{\mu g}{T_0}} \cosh \frac{\mu g}{T_0} x + C_1$$

$\underline{C_1 \to 0}$: → boundary condition.

Then we get,

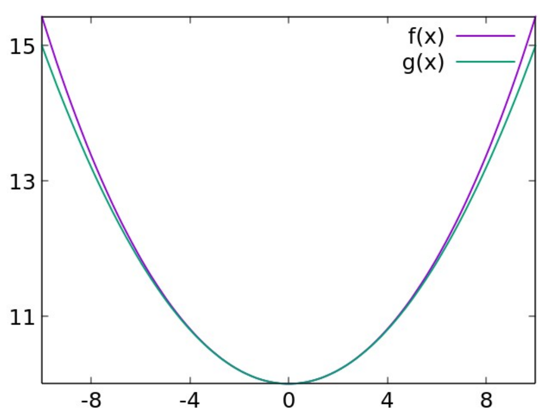
$$\boxed{y(x) = \frac{1}{\frac{\mu g}{T_0}} \cosh \frac{\mu g}{T_0} x}$$

So this is the equation that determines the shape of a massive cable without any other extra load.

So the question is that this equation looks very different from the parabolic equation that we found for a massive massless cable, but let us see whether this is really different or not. So, here I show two functions f(x) and g(x) are shown.

(1) $y(x) = 10 \cosh (x / 10)$.
(2) $y(x) = 10 [1 + 0.5 (x/10)^2]$

Q: Which one is which?

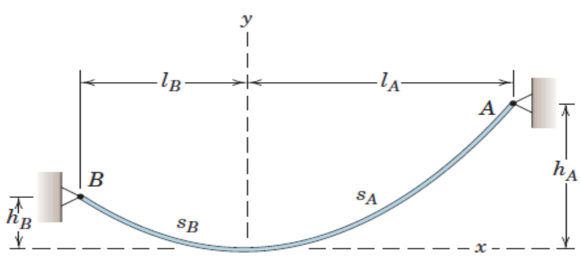


One of these functions is mentioned here, So one function represents this cos hyperbolic function, and the other function represents the parabolic function; these are the functions chosen so that it matches the equation that we found from our analysis, So, this will represent a massive rope or cable and this will represent a massless rope or cable. Now, the question for you is that, but I do not tell you which function is this cos hyperbolic and which function is parabolic, the point is that if you look at so but here I am sort of showing you the plots and if you look at the two curves you will see up to quite a short distance away from your origin that is the lowest point of the hanging cable for a small value of the distance away from the lowest point, these two curves are practically indistinguishable; it is only when you are going at a large length that these two curve shape of these two curves start to differ.

So, by the way, the shape of this curve, the cos hyperbolic curve, which represents the shape of a rope which is hanging under its own weight, is called a catenary. Now, I will give you as a take-home exercise to sort of plot these two function in your favourite plotting tool, and then you determine which curve represents the massive rope and which curve represents the massless rope.

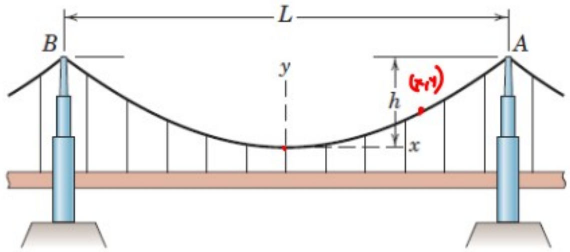
So, the point is for distance, which is about more than certain, let us say 4 unit of length. Then you will start to see that the shapes are not exactly same but are close to origin. You can sort of assume that it is approximately a parabola. So, this is the reason for a long time, it was not recognized that the shape of a massive rope is, in fact, different from the shape of a massless rope.

Now, we take a kind of some application-oriented calculation. So, for example, a simple application could be that if you want to design a suspension bridge and here your material is usually the constraint, so your length of the rope is given. Then, what will you choose as a distance between the two supports? Where are you going to choose the distance between A and B?



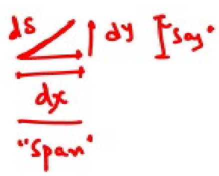
I will explain this picture. So, this is, again, let us take this origin from here to one end. This is called the span of the cable or the span of the bridge and then the vertical height of the support point. So, this represents how much depression from the support the cable has. So this is called the sag. Similarly, for the other end from here to here is your span on the other end, left-hand end, and this is the sag on the left-hand end.

So if you add this distance, which is l_B, and this distance l_A, which gives you the total horizontal span of your bridge, let us say similarly. So now the point question is that the length of the cable is given, then what will be the distance between A and B? So, in general, to answer this sort of question, what we require to sort of know a relation between this span and the sag and the other possibility could be that what is the material that you are going to choose for this suspension bridge and for that, you need to determine the tension that is there along the length of the cable. So, here, the problem will be to sort of know the tension, so at each point, if this is the origin, this is the x-axis, y-axis, each point has a coordinate x and y, so determine the tension at this point (x,y).



So, in this particular picture, I have taken a symmetrical hanging bridge. Above was an asymmetrical hanging bridge, which was the more general case. This is a symmetrical hanging bridge, so l_A is equal to l_B, and h_A is equal to h_B. So, I am not going to do a full analysis of this kind of problem because that is will take it is not the focus of physics.

I am just sort of giving you the fundamental physical relation. You do not have to remember too much. You can start from the first principles and then proceed with analysis in those cases. So, suppose you take a small piece of rope which has length, let's say ds now it has a from one end to another end, the horizontal distance is dx, and the vertical distance is dy, and then this will be the horizontal span so this will be called span of this small piece, and this will be called the sag of this small piece.



So, you want to know the relation between span and sag so that is given by, for example- we have this relation that from geometry we can write this from, and then if I take dx out, then I can write it as, so this is the kind of mathematical relation between this span and sag of a small piece of rope.

$$ds = \sqrt{dx^2+dy^2}$$
$$= dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Now, if you integrate this, let's say from this end 0 to this A, then you will get on the left and integrate on both sides and then it will go from 0 to l_A. So, s cross from length 0 to s_A. So, this is the kind of general relation, so this will give you s_A.

$$\int_0^A ds = \int_0^{l_A} dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\boxed{S_A = \int_0^{x_A} dx \sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

So, this is the general relation that you get. So, now in this case, you can do a further simplification so, if I will put it as a take-home exercise. So, in the case of mass-less rope, in this case, we derived that your dy/dx was some simple function. Now you can put it here and then assume that then you can expand this so that, dy/dx<1.

So you can do a series expansion, and then you get represent a relation with, you can represent and see what happens. Now, I will quickly go through the other case. So again, it is basically similar exercise.

So now, if you want to know the magnitude of tension at each point. So this will be given by the our remember our vector, so it is a vector rule. So, this force has two components, x, and y,

$$\frac{T_y}{T_y} = \tan\theta = \frac{dy}{dx}$$

Massless rope

$$T(x,y) = \sqrt{T_x^2 + T_y^2}$$

$$= T_x \sqrt{1+\left(\frac{dy}{dx}\right)^2} \rightarrow T_x \sqrt{1+\left(\frac{W_0 x}{T_0}\right)^2}$$

$$T(x) = T_0 \sqrt{1+\left(\frac{W_0 x}{T_0}\right)^2}$$

Now again, if you for the case of massless rope, this is a simple, simpler function, which we derived in the last previous couple of lectures. Now I could give you another exercise. So now T_0 was some constant this was a constant, so now you have this function, so assume some value of T_0, and you plot this function plot as a function of x. So, determine where this function has this tension has the highest value.

So, just from physical intuition, if you look at this picture, you see that if you start from origin and go here as you go towards right-hand side so, tension at this point, for example, it has to support the weight of the total length of this rope now as you go and this point it has to support the weight of the total length from 0 origin to up to this part of the rope. So, you can expect that the tension must be highest at the endpoint. So, you can plot this and verify whether this, as per expectation, matches your plot. So, thank you, and in the next few lectures, we are going to take more interesting examples of the static condition, but before that, let me summarize like how we analyze this problem. So we include it in three steps; first, you choose the system, which, in our case, in this example, we choose a small piece of cable at the point (x,y), then you identify all interactions acting on the system. So, this is where your Newton's laws of motion, third law, and second law will come in handy, and then our third part is the analysis.

So, in this case, the tool that we used was the force balance. And from applying this force balance, we got a differential equation for the local curvature, which means the second derivative of y as a function of x. The y present in the curve of the rope and then we impose the boundary conditions is very crucial to solve this differential equation and determine the shape of the rope. And also, we saw that this slope the local slope of this rope is important to determine the tension as well as to get a relation between the span and sag. So, this concludes our discussion on this suspension cable suspension rope problem. We will take more examples in the next few lectures. Thank you.