Newtonian Mechanics With Examples

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Week -03

Lecture -12

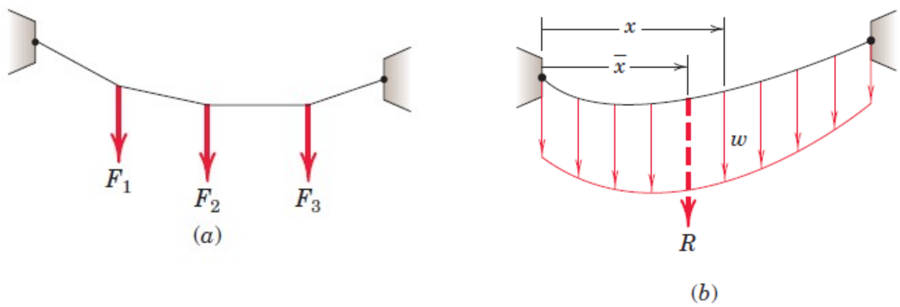
So, let us continue our discussion on mechanical equilibrium. So, in the previous lecture, we started discussing a problem of cables, and we discussed little bit about what are they, so the nature of the tension force, and we reviewed critically about three particular assumptions that and we are going to now do a problem where we will use those assumptions. So, the point here is that the normal assumptions that you make about rope and tension force in textbook problems, you have to see whether they are really valid in real-life situations. So, let us start by defining our question. So, we are taking a class of problem which are called suspension cable. Now, this is a vast topic and those of you who are taking various engineering courses, such as mechanical or civil engineering, you will learn a lot about this problem in your respective engineering course.

From physics point of view, so, I am going to pick up a selected subset of questions and this is what I want to discuss. So, imagine that you have a cable which is hanging between two ropes two ends such as for example, let us say this is a cable which is hanging between two supports, so two supports which are fixed. Now, what we know are the external forces. So, imagine that you are hanging clothes, for example, on a rope, or it could be some suspension bridge.

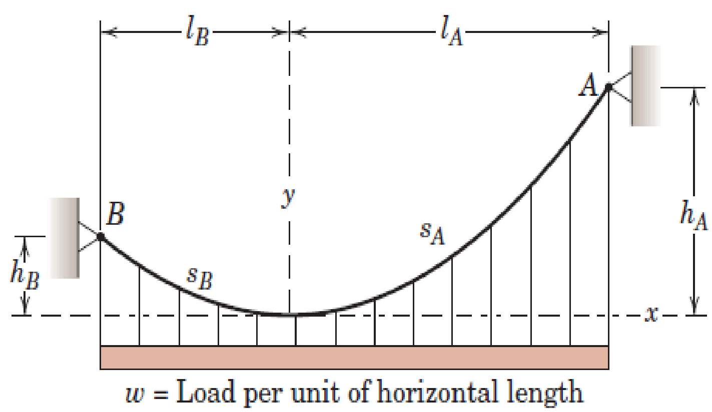
So, there are external forces on the loop, and you know those forces. You also know the end-to-end length, so this length from this end, suppose this is the end, this is the end, you also know the end-to-end length, and you also know what are the force boundary conditions. So, we will clarify what the boundary condition very soon. So, basically, you know what are the details of at the end, the forces acting on the end, for example. And then we are interested to know about some simple questions like, for example, what is going to be the shape of this rope if it is supported hanging between two fixed ends.

So, which means that, for example, What will be the curvature? What which so the local curvature determines the shape of this rope. And we will also discuss something called sag and span. And the other practical question that may be interesting from, especially from the engineering point of view, is what is the tension in the rope. Now, I will say a little bit about the external loads. So, in general, they can be point load which are applied at discrete points, such as there are three loads, three for external forces, vertical forces

F1, F2, and F3 in this picture (a), or there could be a load which is applied continuously along the length of the cable or the rope (b).

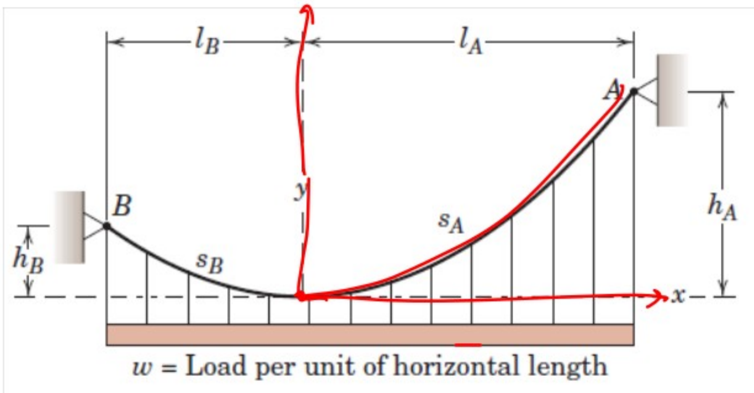


So, in this particular problem, we are going to assume that this is the case. So, now let us define the example more concretely. So, as I mentioned so these are two ends which are fixed and these two ends, A and B, and there is a cable whose length is l which is hanging between two ends. Note that these two ends, in general, can be at different heights. So, they need not be symmetric.



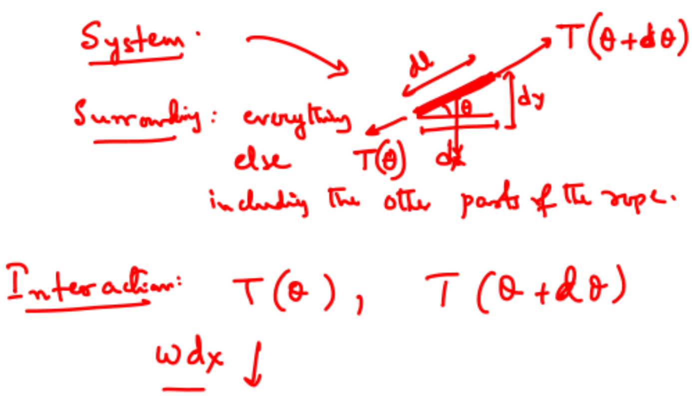
So, a flexible and we discussed why do we need flexible weight, a flexible cable in the previous lecture, and we are going to assume in this particular example that the cable is massless. So, one of the goals of this problem is to study the effect of the mass of the cable in the analysis. So, in this example, we are going to study a massless cable, which is simpler, and in the next example, we are going to add the mass of the cable in the analysis. Now, there is an external vertical load w per unit horizontal length. So, this is a continuously applied length load, which is, let us say, represented by this horizontal red line, which is applied everywhere, and the amount of the load this is of load means of force is w per unit length.

Now, it is given that the structure is static. It is in equilibrium. Then, the problem is to determine the shape of the cable. So, if this is my suppose, this is my origin and I draw an horizontal line, which is my take my x-axis, and then I take this is my y-axis, then this cable makes a curve which is y as a function of x in space. So, the question asks that, what is this function y as a function of x? Now, one thing I want to emphasize here is that the note the way I have chosen the coordinate system. This is not random.



So, I have chosen the lowest point of the row as my origin, and then I have chosen the tangent to the lowest point as my x-axis and the perpendicular direction as my y-axis. So, this is a deliberate choice to make the shape very simple and the equation of the shape as simpler as possible. And as you can expect, in general that if you shift the origin, then the equation of the curve will shift will also change. So, let us try to analyze. So, first thing is what is going to be our system.

So, I am going to take our system as a piece of the rope. Let us say this is my rope or cable with some so it has is a small piece. So, we expect that we are anticipating that the answer is given by some sort of a differential equation. So, we are going to take a small piece. So, let us say the length is dx, the horizontal length is dx, the vertical length is dy, and this length is dl.



So, it is a small piece of rope, and this makes, this piece makes an angle θ with the horizontal if the x axis. Then this is the tension. So, this is my system and everything else is surrounding is my system and everything else is surrounding which includes the other parts of the rope. So, now we define the system and chosen the system and surrounding. Now, we look at the interactions.

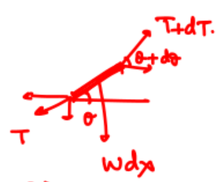
What are the forces? Now, the system is interacting with the neighboring part of the rope. So, there is a tension force on this side, and it is also interacting with the contact force with the neighboring part on the left side. So, there is also a tension force. And in the last lecture, we saw that the direction of the tension force is known. It must be along the tangent of the of the of the rope. However, since this system is not horizontal, it is the tangent direction is continuously changing. So, these two forces are not in the same direction, which means they are not the same force.

So, let us call that force T(θ), and then at this point at this end, the angle with the horizontal is no longer θ, but T(θ+Δθ) or T(θ+dθ). This is crucial. Now, why are they different? You can easily see that because the so this force on the this so the step the reason the if you look at the force balance of this slice of the rope here. Now, there is another force, so these are T(θ) and T(θ +dθ), as shown in the picture, and also the vertical force. So, there is a vertical force which per unit length, so this is a vertical force in the downward direction.

So, that means that if you if we look at the force on the right-hand side, we must balance not only this tension force on the left-hand side but also the vertical force, and that is what makes the tension non-uniform in this case. Remember our discussion from the last lecture. So, these are the interactions. There is no other interactions, so I am going to assume that the rope is mass less. So, there is no interaction between our system and the Earth. So, these are only all the forces we have listed.

So, now the whole system is in mechanical equilibrium, so the third step is to write down the condition of force balance, the mechanical equilibrium. So let us say first we look at the force, so let me draw the picture again at this end: this is θ, this is dθ, this is (θ +dθ), there is a force w dx. Note that this w is a load per unit horizontal length. This is a crucial point we will come to that. So, then, what is the total force on the horizontal direction? So, this force has a component, so this is (θ +dθ), so there is a force T cos (θ +dθ) minus T cos θ.

Force balance.



(1) $\sum F_x = (T+dT)\cos(\theta+d\theta) - T\cos\theta = 0$

(2) $\sum F_y = (T+dT)\sin(\theta+d\theta) - T\sin\theta - w\,dx = 0$

(1'): $(T+dT)(\cos\theta \underbrace{\cos d\theta}_{\approx 1} - \sin\theta \underbrace{\sin d\theta}_{\approx d\theta}) - T\cos\theta = 0$

$(T+dT)(\cos\theta - \sin\theta\, d\theta) - T\cos\theta = 0$

Now, w is a vertical force; it does not have a horizontal coordinate component, and this must be 0 that is the condition of force balance. Similarly, let us write down the condition of let us call that equation 1, and then equation 2 is the force balance in the vertical direction, so we have the sine component.

$$\left[dT\,d\theta \approx 0\right] \qquad dT\cos\theta - T\sin\theta\, d\theta = 0$$

$$d(T\cos\theta) = 0$$

$$\Rightarrow \quad T\cos\theta = \text{const.} = T_0$$

So, let me Now simplify the equation 1. There are product of dT dθ so that is a second-order difference product of two infinitesimal so we are going to assume that that will be 0 because it is a product of two infinitesimal. So then we can write the left-hand side as Now, note that this I can, it is same as saying that the differential of T cos θ is 0, that means that T cos θ is must be a constant. Let us call that T_0 .

Now, let us take the vertical equation.

$$②:\ \Sigma F_y = (T+dT)\sin(\theta+d\theta) - T\sin\theta - w\,dx = 0$$

$$\left.\begin{array}{l} \cos d\theta \sim 1 \\ \sin d\theta \sim d\theta \\ dT.d\theta \sim 0 \end{array}\right]$$

$$d(T\sin\theta) = w\,dx \quad ③$$

$$\frac{d}{dx}(T\sin\theta) = w \equiv w_0 \ (\text{constant})$$

w can be $w(x)$

Now let us reflect what we have done so far. We have applied the force balance conditions, and we got, let us call that equation, 3, and we got this condition. Now, in the given problem, it is given that this w is constant. So, to emphasize that, let us call that w_0, but note that in general, w can be a function of x, but in this problem, it is given that w not is constant. It is a constant force load applied per unit length.

Now, the other thing is that, again, in this problem, we basically have derived two equations. One equation is from the force balance condition from which we get that the horizontal component of the tension force is constant. And we get an another equation which is about the vertical component of the tension force, which is equal to w_0. But so, in this case, there are three unknowns. So T is an unknown, and we also have θ, which is an unknown.

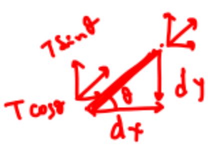
So the problem asks you to calculate the y as a function of x, which represents the shape of the cable. So, we need to eliminate θ. and replace it by y. Now, how do we do that? So note that there is a very simple, I mean there is a simple geometric insight. So if I look at this piece of the cable and this has a length, which is dx, and it has an elevation, which is dy, and this is the angle θ.

So then, from geometry, we see that tan θ=dy/dx. The other thing is that the forces are along the tangent, the tension force is along the tangent. The horizontal component is, let us say at this end, this is T cos θ, and this is T sin θ. So tan θ can be written as So this is the crucial fact that the tension force is along the tangent. We can exploit this fact to replace eliminate θ.

So how do we do that? So, this is now the vertical component. So what we have got is that the horizontal component is the, there is a differential equation in terms of in with respect to the vertical component of tension. So you take this equation, this one, and take a derivative with respect to x. Let us do it here. So we have this T sin θ divided by T_0 .

$y(x)$ Eliminate θ

$$\boxed{\begin{aligned} d(T\cos\theta) &= 0 \\ \frac{d}{dx}(T\sin\theta) &= w_0 \end{aligned}} \rightarrow T\cos\theta = T_0$$



$$\tan\theta = \frac{dy}{dx} \qquad \frac{d}{dx}\left(\frac{T\sin\theta}{T_0}\right)$$

Tension force along the tangent:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{T\sin\theta}{T\cos\theta} = \frac{T\sin\theta}{T_0}$$

Now, if I take a derivative of this quantity, so we get -

$$\tan\theta = \frac{dy}{dx} = \frac{T\sin\theta}{T_0}$$

derivate on both sides w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{T\sin\theta}{T_0}\right) = \frac{w_0}{T_0}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{w_0}{T_0}} \Rightarrow \frac{dy}{dx} = \frac{w_0}{T_0}x + c$$

$\Rightarrow c = 0$

So we get a differential equation, now we can solve it to determine y is a function of x.

Boundary condition

at $x = 0$, $\begin{cases} y = 0 \\ \frac{dy}{dx} = 0 \end{cases}$

$w_0 = 0 \quad y = 0$

$$\frac{dy}{dx} = \frac{w_0 x}{T_0}$$

$$y = \frac{1}{2} \frac{w_0}{T_0} x^2 + C_1 \Rightarrow C_1 = 0$$

$$\boxed{y = \frac{1}{2} \frac{w_0}{T_0} x^2}$$

So let us do that. So first, let us take one. Now, here is where, as you know, that in order to get a particular solution of a differential equation, you need some conditions. So here is where you need the boundary condition because there will be constants of integration, and to determine those constant of integration, you need some conditions. So, we are going to assume that this is where our choice of reference system comes in handy.

So this is the shape of a mass-less cable which is hanging because of some external load and supported at both end. Now, once you have some relation, the first thing to do is to check whether this relation makes sense. So here is a simple change, and for that, you need to take some simple limiting case and see whether this makes sense. So the simple limiting, what could be a simple limiting case, suppose that there is no external force, that is $w_0=0$, the load per unit horizontal length is 0. In that case, your equation is just y=0.

So, in that case, we expect that the rope is a mass-less rope, so there is no, absolutely no horizontal force. Then you expect that the rope should be perfectly horizontal, and we indeed guessed a solution y=0, which represents the x-axis. So our solution indeed is matching with our expectation. So, this solution, which is a parabola, makes sense. Our next goal is to now we are going to put the effect of mass; we will going to include the mass in our calculation and see what happens.

So that will be the topic for the next lecture. Thank you.