Applied Optics
Professor Akhilesh Kumar Mishra
Department of Physics
Indian Institute of Technology, Roorkee
Lecture 08
Thick and Thin Lenses, Unit Planes

Hello everyone, welcome to my class we were learning about Matrix Method in Geometrical Optics. We have already talked a lot about matrix method and we saw how to represent translation in form of a matrix and then we learn how to represent refraction in form of a matrix. And then for a complex optical system or for some given optical system, which is like a black box, we try to form our matrix for this black box and then we learn that there is only one unknown element. We will implement the knowledge of our matrix to many different system.

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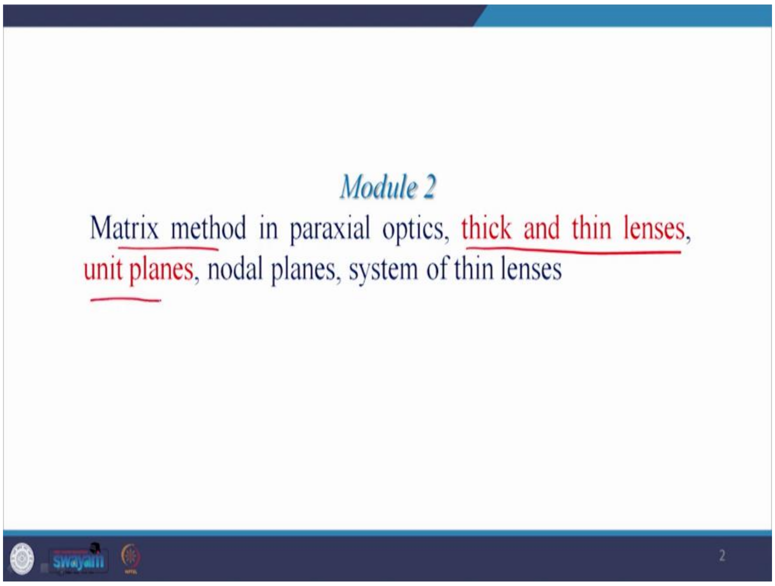

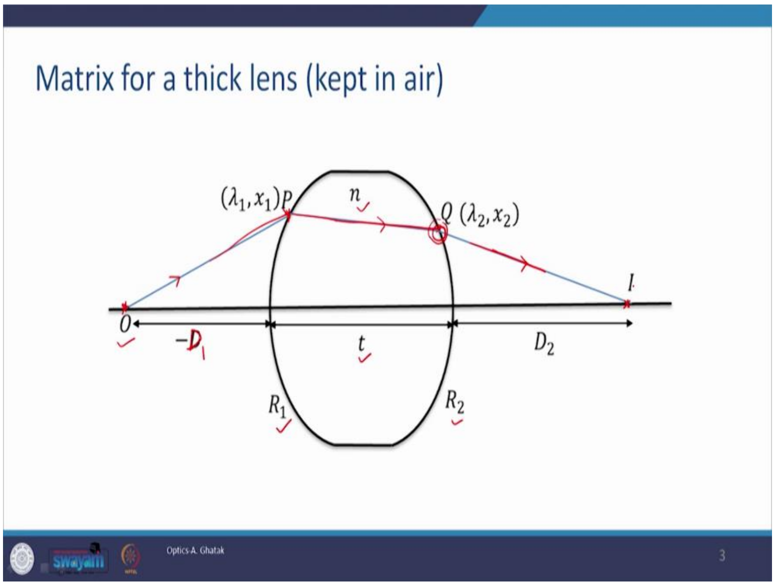

And today, we will learn to implement the knowledge of our matrix method for thick and thin lenses, we will analyze thick and thin lenses in the purview of matrix and thereafter, we will talk about unit planes. To start with let us suppose that we have a thick lens which is kept in air. And the thickness of this lens is given by t, the refractive index of the lens material is n and since this lens is kept in the air, the refractive index outside this lens is one.

Now, suppose an object O is kept at a distance D_1 from the left refracting surface of this thick lens while image I forms at a distance D_2 from the right refracting surface of this thick lens. Here R_1 and R_2 are the radii of curvature of left and right refracting surfaces of this lens.

Now, suppose from point object O a ray emanates and falls at point P on the left refracting surface and then it undergoes translation within this thick lens and then again it refract at point Q and then it again undergoes translation and reaches at point I. O to P, P to Q and Q to I are translation while at point P and Q the ray undergoes refraction, the coordinates of point P and Q are (λ_1, x_1) and (λ_2, x_2) respectively.

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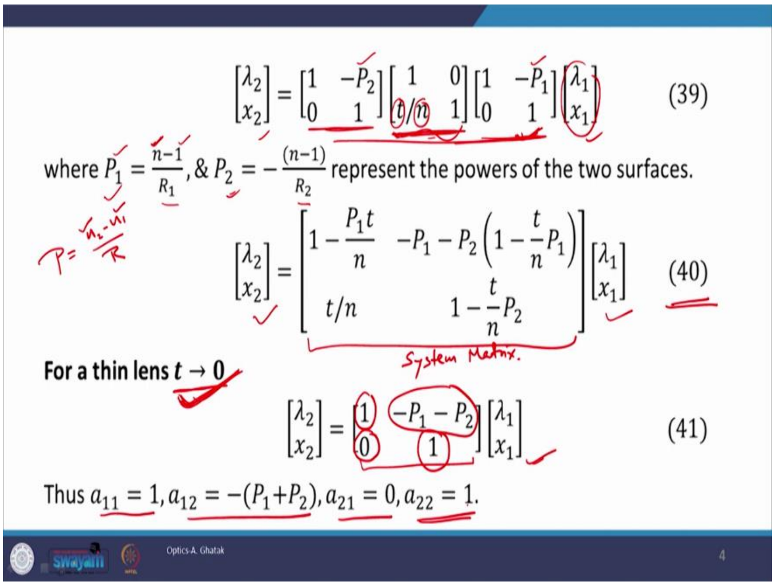

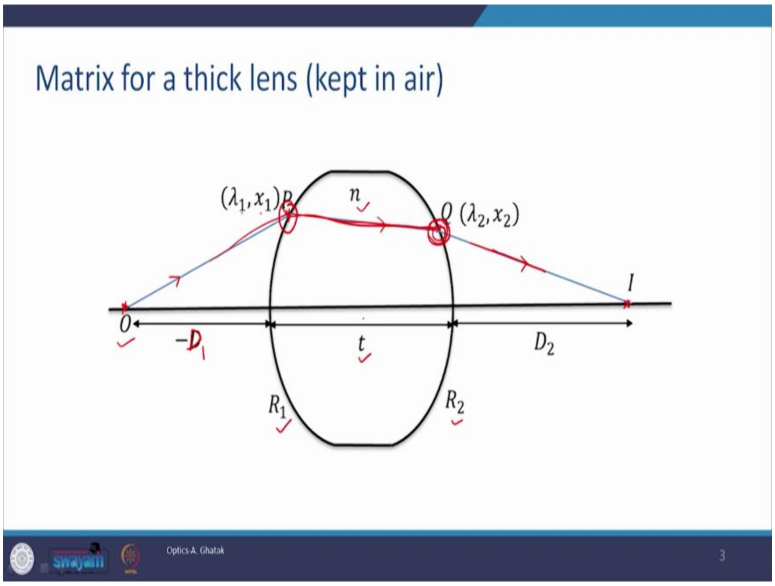

Now, to analyze it, we will try to relate the coordinates of point P and Q. P has the coordinate (λ_1, x_1) and Q (λ_2, x_2), how to relate them, we will use again the matrix formulation which we learned from our previous classes we will multiply (λ_1, x_1) with refraction matrix. And this refraction happens at point P, here (λ_1, x_1) is the point coordinates of point P and (λ_1, x_1) is the coordinate of a point which is just on the top of the refracting surface.

And at this point refraction happened therefore, we multiply (λ_1, x_1) with the refraction matrix, right after refraction the ray travel within the thickness of the lens therefore, we will multiply with the translation matrix, the second matrix is the translation matrix, where t is its thickness and n is the refractive index of the lens material and at last at point Q the ray again suffers refraction.

Therefore, we again multiply the refraction matrix, here P_2 and P_1 are the power of refracting surfaces. P_2 is the power of right refracting surface while P_1 is the power of left refracting surface and the expressions for P_1 and P_2 are given here, n is the refractive index after lens material and 1 is the refractive index of the material outside or the medium in which the lens is kept.

And in our case lens is kept in air we know that the expression for power P=$(n_2 - n_1)/R$, n_2 is the refractive index of the second medium, n_1 is the refractive index of the first medium. Therefore, for P_1, since the second medium is glass therefore, instead of n_2 we will write n and the first medium is air therefore, instead of n_1 we will write 1 here. Similarly, for P_2 the relation would be minus $-(n-1)/R_2$. These are the powers of the two surfaces.

After substituting these and multiplying all these three matrices we will get this expression for system matrix, this is our system matrix, system matrix relates point P with point Q. Now, this system matrix is for a thick lens. Let us go to a limiting case where our lens is thin what do I mean by thin, thin means we can equate thickness t to 0, equating thickness t to 0 means this thickness t is very small as compared to the height of the lens, the distances of the objects and image from the lens and any other related distances which are involved in the calculation.

We know that all the distance, if all the distances are very large as compared to the thickness of the lens then we can apply this condition then we can use the t is almost equal to 0 and if we substitute t is equal to 0 in equation 40 then we will get equation number 41 and then our system matrix is now reduced, it gets simplified. The first term of the system matrix which is a_{11} earlier is equal to 1 now, the second element of the system matrix which is a_{12} is now equal to $-(P_1 + P_2)$, the third element which is a_{21} is equal to 0 while the fourth element which is a_{22} is equal to 1. a_{11}, a_{12}, a_{21} and a_{22} these are the elements of the matrix which we have already defined while studying in the last class.

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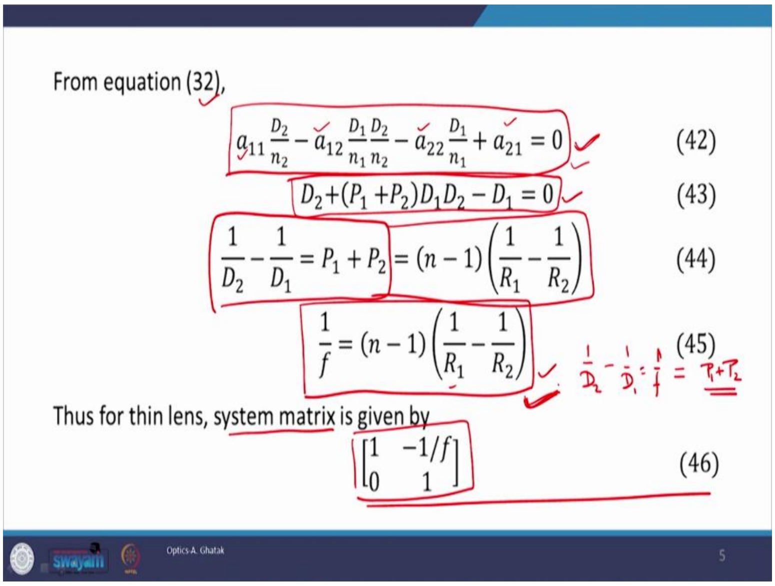

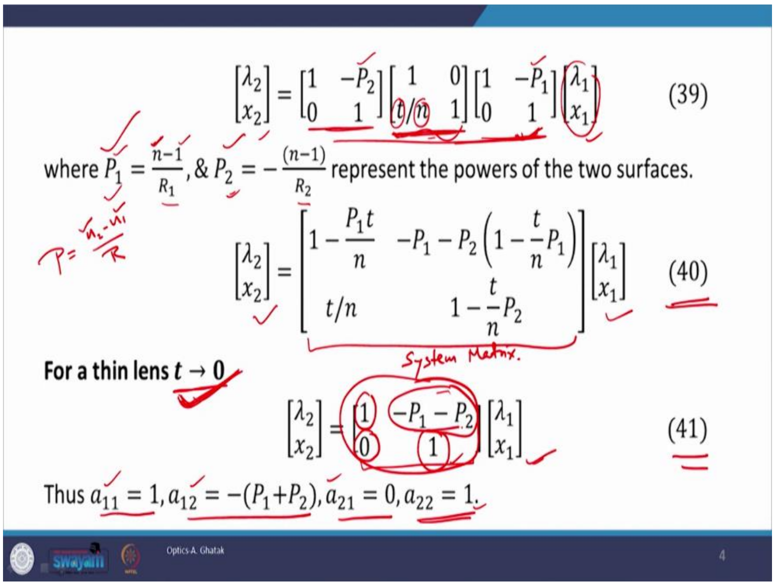

Now, from equation 32 of previous class where we just equated the third term a_{21} term to 0, we get this here, this is exactly the equation number 32 of previous class. Now, we will use this relation, we will use this expression, how would we use this, we will substitute for a_{11}, a_{12}, a_{22} and a_{21} and after this substitution what are these, now we know that a_{11}, a_{21}, a_{22}, we have the expression of all these a's, we will substitute for all these a in equation 42 and this will give equation number 43.

Now, if we simplify 43 then we will get this relation, $1/D_2 - 1/D_1 = P_1 + P_2$, but we already have the expressions for P_1 and P_2 here, we will substitute for P_1 and P_2 and this gives this relation which is $(n-1)(1/R_1 - 1/R_2)$ which we are already familiar with this is nothing but the expression for inverse of focal length, 1/f $=(n-1)(1/R_1 - 1/R_2)$ this we have already derived this is focal length. Therefore, we can write $(1/D_2 - 1/D_1) = 1/f$.

Now, let us go to the equation number 41. We know that the system matrix here is 1, 0 and then $-(P_1 + P_2)$ and then 1. This is our system matrix here in equation 41 for thin lens. But we know that $(1/D_2 - 1/D_1) = 1/f = (P_1 + P_2)$ therefore in our system matrix we know that $(P_1 + P_2) = 1/f$ and therefore, our system matrix modifies to equation number 46. This is our system matrix now, the second element is minus of 1/f, inverse of focal length.

And for thin lens we have already derived equation number 45 and using the matrix formulation we see that we again get the same relation, we again get the same formula, it means that we are on the right track and we are not using Snell's law here, it is a very easy now. Now, this all work for thin lens, what will happen for a thick lens?

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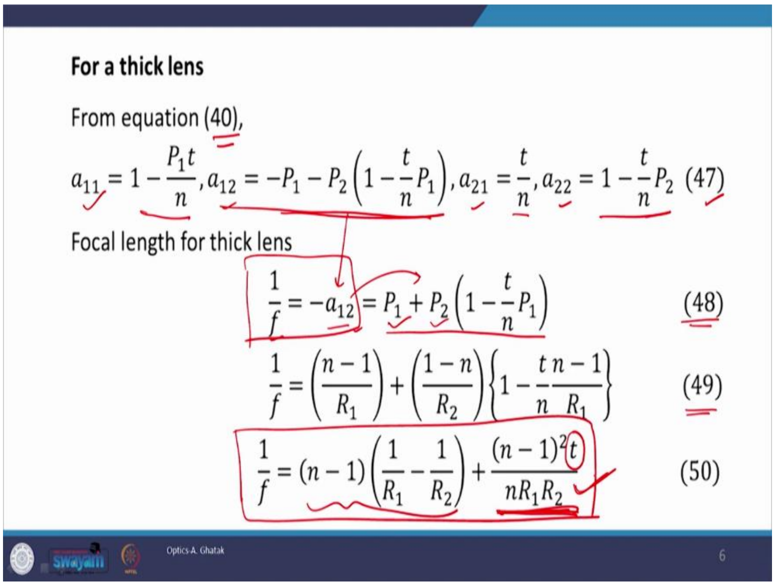

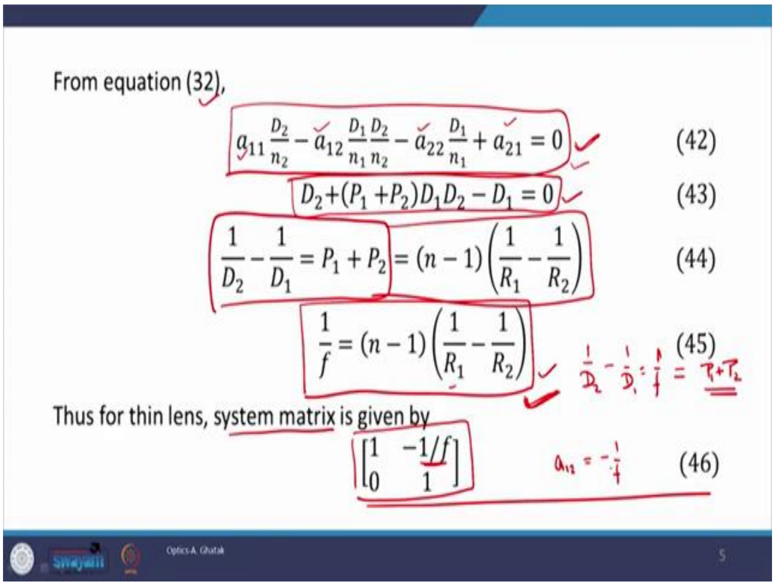

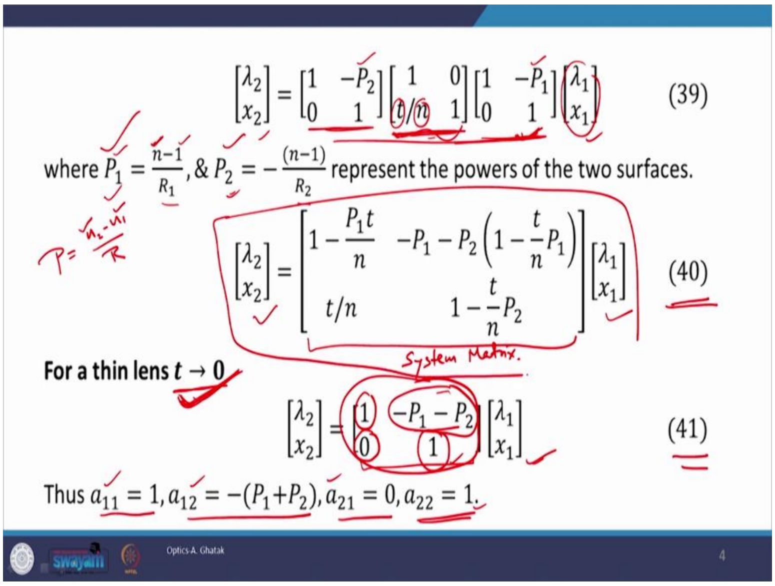

We will again go to equation number 40. What was equation number 40? This is our equation number 40, the master equation. Now for thick lens, we cannot equate t is equal to 0 then we will see what are the matrix element here. The first element a_{11} is equal to this, the second element is this, third element is this and the fourth element is this. This is our equation number 47. These are the elements matrix elements.

For thin lens what we found is that the second term which is $a_{12} = -1/f$ or $-1/f = a_{12}$, but we will see while like a thick lens can easily be analyzed, can easily be studied using unit plane, the concept of unit planes would be explained or addressed in the next topic.

But for the thick lens to this formula holds good and therefore, the expression for inverse of focal length for thick lens can be given by equation number 48, this relation 1/f is equal to minus of a_{12} also holds good for a thick lens and what is the expression of a_{12} this is the expression of a_{12} when we will substitute it back here and then we will get this relation. Now in this equation 48 if you substitute for P_1 and P_2 then we will get equation number 49. And if you simplify 49 a bit then this is the final expression for focal length which you get.

Now, you see that in addition to the term which we saw case of thin lenses there is this extra term and this term appears only when we take into account the thickness of the lens t whenever t is nonzero then this extra term the second terms appear in the expression of focal length of a lens.

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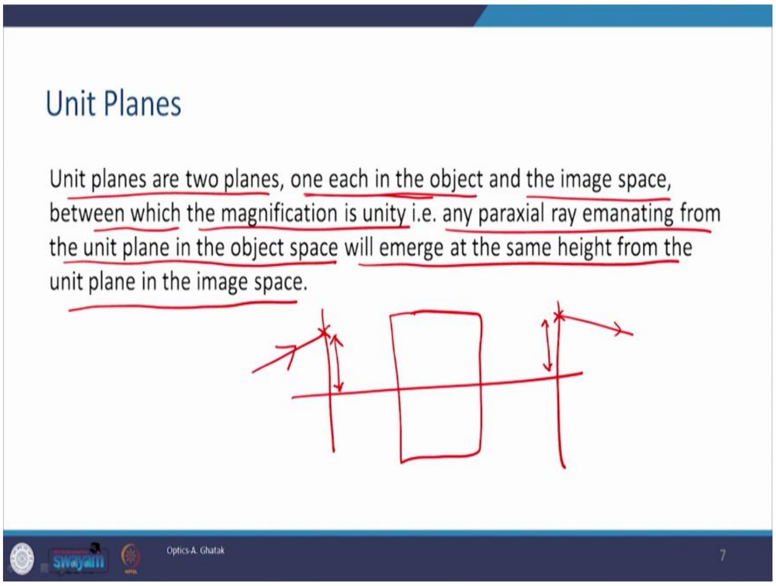

Now let us introduce the concept of unit planes, what are the unit planes? Unit planes are two planes, one each in the object and the image spaces. There is a lens system left to the lens system is objective space and right to the lens system is your image space. Now, what the statement says is that there is a one pair of unit plane, here there are two planes, one is in the object space while the second is in the image space. The condition for these two planes to be unit plane is that the magnification must be unity.

I repeat, unit planes are two planes when each in the object and the image space between which the magnification is unity, the magnification which we already studied about this must be unity for the unit planes. What does it mean? It means that any paraxial ray emanating from the unit plane in the object space will emerge at the same height found the unit plane in the image space.

Suppose this is our lens system. In this box the lens system is there. And this is the unit plane in the object space and this is the unit plane in the image space. Then this statement says that if a ray falls on the first unit plane then it will emerge out of the second unit plane from the same height, this height must be equal to this height. This is how the unit planes are defined, unit plane must have unit magnification, the magnification must be unity.

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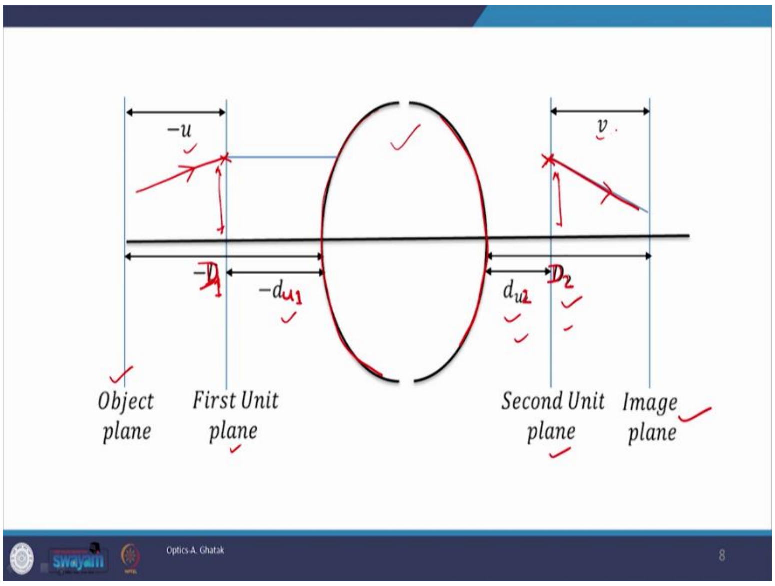

Now, the schematically the diagram is shown here. Now, this is our lens system and the left sphere represents the left refracting surface and this represent the right refracting surface. This object plane is given here and it is given by at the object plane is at a distance D_1 from the left refracting surface and the image plane is at a distance D_2 from the right refracting surface. The first unit plane which is in the object space is at a distance d_{u1} from the left refracting surface while the image plane the second unit plane exists at a distance d_{u2} from the right refracting surface.

I repeat D_1 is the distance of object plane from the left refracting surface while D_2 is the distance of the image plane from the right refracting surface and similarly d_{u1} and d_{u2} are the distances of first and second unit planes from the left and right refracting surface of the optical system respectively. All the distances on the left are measured in negative therefore, these distances are negative while the distances on the right hand side are positive.

Now a ray here will emerge from the same point, if you launch a ray at the first unit plan then it will emerge from the second unit and from the same height. This is the basic definition of unit plane and u and v are the distances of object and image plane from the first and second unit planes respectively this is all about this figure, but let us go to the analysis domain.

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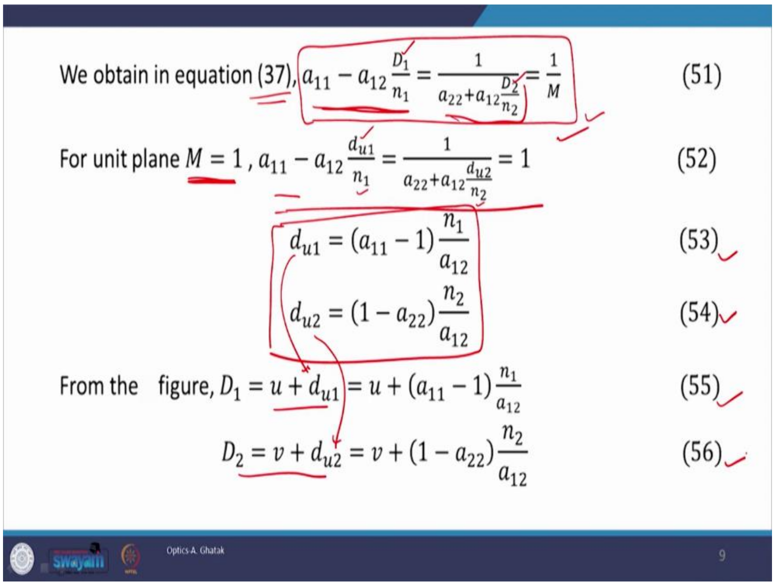

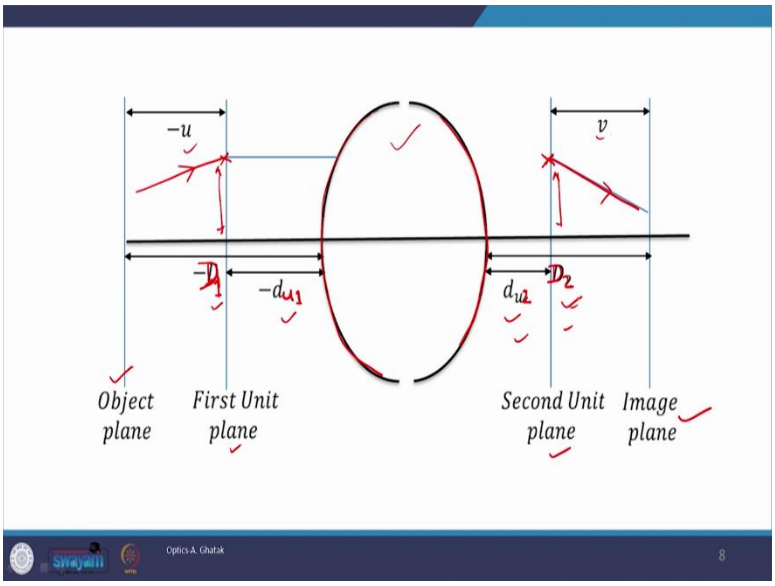

This we have already done, this calculation we have already derived while studying the black box system here the optical system within the black box, we have a number of lenses where they are with a certain separation and therefore, we were not knowing the matrix for this black box and the matrix element we assumed as a_{11}, a_{12}, a_{21} and a_{22} and then finally, we got that except for one matrix element, all the elements can be calculated using the matrix representation of lenses.

And during that derivation, we found that this expression is equal to inverse of this expression and which is equal to inverse of magnification. Now as per the definition of unit planes, the magnification must be equal to 1, unit plane has unit magnification therefore, if we equate M to 1, then equation number 51 gives this relation, where D_1 and D_2 is replaced by respectively d_{u1} and d_{u2} which are the distances of unit planes from the lens system. And n_1 and n_2 as we know they are the refractive index of the media on the left and right hand side of the lens system.

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From equation (32), $a_{11}\frac{D_2}{n_2} - a_{12}\frac{D_1}{n_1}\frac{D_2}{n_2} - a_{22}\frac{D_1}{n_1} + a_{21} = 0$ (57)

$$D_2 = \frac{a_{22}\frac{D_1}{n_1} - a_{21}}{\frac{a_{11}}{n_2} - \frac{a_{22}D_1}{n_1 n_2}} \quad (58)$$

Substituting D_1 and D_2 from equation (55) and (56)

$$v + (1 - a_{22})\frac{n_2}{a_{12}} = \frac{\frac{a_{22}}{n_1}\left[u + (a_{11} - 1)\frac{n_1}{a_{12}}\right] - a_{21}}{\frac{a_{11}}{n_2} - \frac{a_{22}}{n_1 n_2}\left[u + (a_{11} - 1)\frac{n_1}{a_{12}}\right]} \quad (59)$$

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We obtain in equation (37), $a_{11} - a_{12}\frac{D_1}{n_1} = \frac{1}{a_{22} + a_{12}\frac{D_2}{n_2}} = \frac{1}{M}$ (51)

For unit plane $M = 1$, $a_{11} - a_{12}\frac{d_{u1}}{n_1} = \frac{1}{a_{22} + a_{12}\frac{d_{u2}}{n_2}} = 1$ (52)

$$d_{u1} = (a_{11} - 1)\frac{n_1}{a_{12}} \quad (53)$$

$$d_{u2} = (1 - a_{22})\frac{n_2}{a_{12}} \quad (54)$$

From the figure, $D_1 = u + d_{u1} = u + (a_{11} - 1)\frac{n_1}{a_{12}}$ (55)

$$D_2 = v + d_{u2} = v + (1 - a_{22})\frac{n_2}{a_{12}} \quad (56)$$

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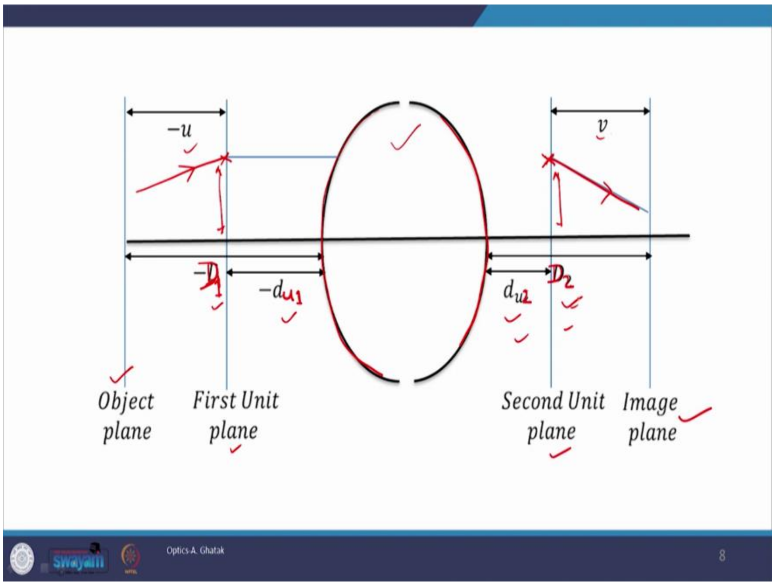

Now, from equation number 52, we can get these two relations. These are the expression for d_{u1} and d_{u2} which are given by equation number 53 and 54 respectively. From the figure $D_1 = u + d_{u1}$ and similarly $D_2 = v + d_{u2}$, this is what is written here. Now in this expression we will substitute for d_{u1} and d_{u2} and then this gives us equation number 55 and 56, equation number 55 and 56 gives the expression for D_1 and D_2.

Now, we will again go to equation number 32 which we derived in our previous classes, which we got after equating the third element of the system matrix, the a_{21} element of the system matrix to 0 here, when we equated the a_{21} element of the system matrix to 0. We got equation number 32 if you just turn your slides and you will see that this is equation number 32. In this equation, which is now written as equation number 57, we see that there are D_1 and D_2 and we will substitute for D_1 and D_2 from equation number 55 and 56 which gives us the expressions for D_1 and D_2 and from here we get this expression. This is a big complex expression. This we got after substitution of D_1 and D_2 in equation number 57 or in equation number 50.

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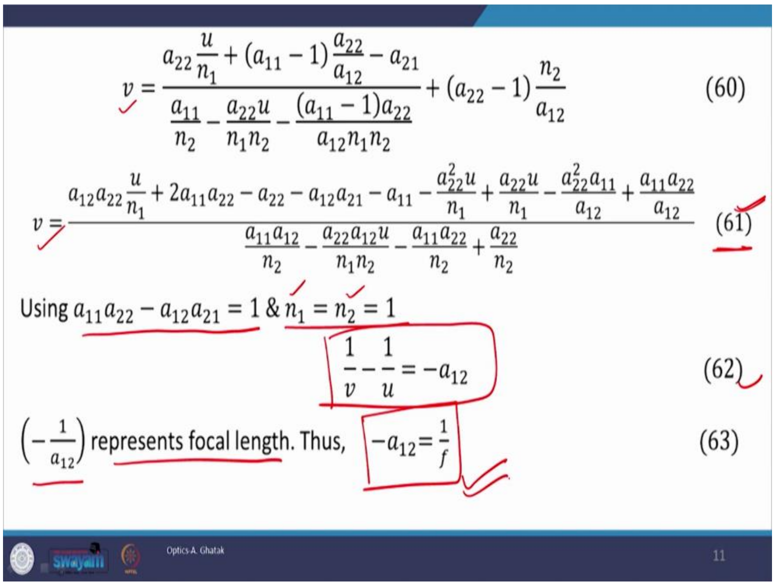

From here we can calculate the expression for v, and after a bit of simplification you get equation number 61 which is very complicated, to simplify it further we will use the property of a system matrix. Now, the important most probability is the determinant of a system matrix must be equal to unity from there we get $a_{11} \times a_{22} - a_{12} \times a_{21} = 1$.

Second simplification arises from this relation, we assume that the lens or the lens system is kept in air therefore, the medium on the left and on the right of the optical system is air therefore, the refractive index would be equal to 1, with these simplifications the 61 reduces to 62 which is 1/v−1/u=$-a_{12}$.

But we know that 1/v−1/u=1/f therefore, $-a_{12} = 1/f$ this is what we use for a thick lens. Therefore, $-a_{12}$ represents focal length and we get this formula. This is the last unknown element of the system matrix for an optical system, unknown optical system. Therefore, matrix method provides us the ways to do this geometrical optics with a lot simplification, with a lot of easiness.

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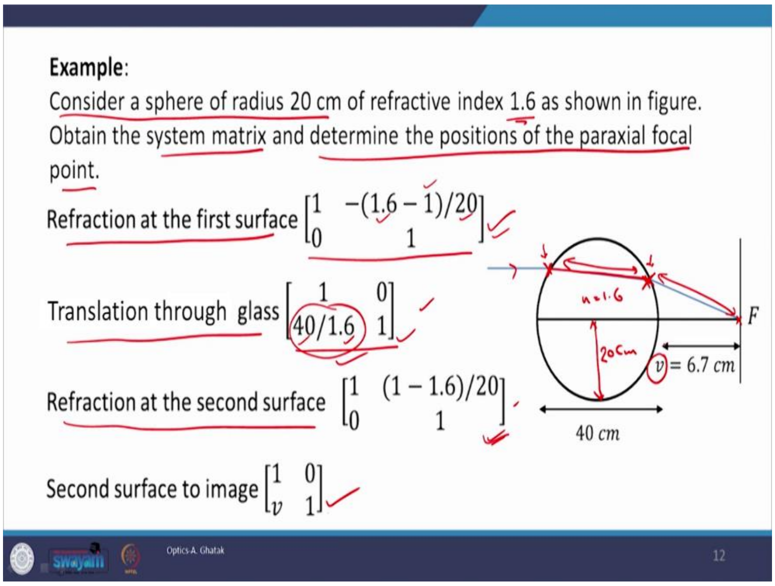

Now, we will take an example and the question is consider a sphere of radius 20 centimeter and refractive index 1.6. The sphere is given here in this figure on right and the radius of this sphere is 20 centimeter and its refractive index is given as 1.6 and the question asks to obtain an expression for system matrix and determine the positions of the paraxial focal point for rays which are close to the axis of the system and which makes very small angle for those rays and the question asked to calculate the focal point.

Now, the ray falling on this point here on this sphere and then it travels a certain distance inside the sphere and then it emanates from this point the right point. And at these two points refraction is happening. And here translation is happening therefore, we will quickly form three matrices, the two for refraction and one for translation. The refraction matrix would be given by this relation we know how to calculate it, it is minus of P and what is P, $(n_2 - n_1)/R$, n_2 is 1.6, n_1 is n and R is 20.

Now, once the refraction matrix at the first point is calculated then we will find the expression for translation matrix. The translation matrix, instead I should call it translation matrix. The translation matrix would be given by this 2 by 2 matrix where this is d/n, d is the distance which it travels and if there is paraxial then this distance will be equal to the diameter of the sphere and 1.6 is the refractive index of this sphere. The second refraction happens here therefore, the second refraction matrix would be given or can easily also be calculated and therefore, all these matrices are known.

But the question says that ray after refraction it converges to focal point, it goes to the focal point therefore, an extra translation is also involved here, therefore effectively we have two translation and two refraction. Therefore, for the second translation, we can write the translation matrix which is given here this so 1, 0 and v; 1 is the fourth matrix element which is the second translation and v here is the distance of the point where it focuses.

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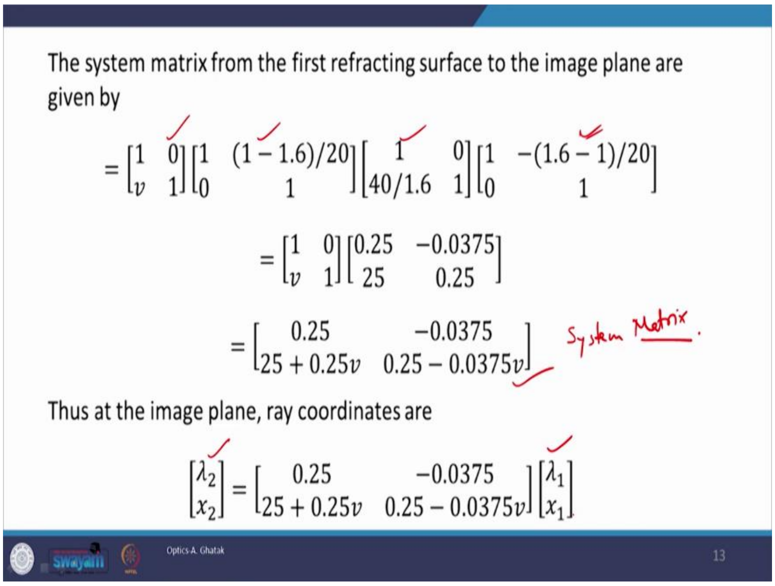

Now, to get the system matrix we will have to multiply all these four matrices as we did earlier, we will multiply these four matrices and after multiplication we will get the system matrix. This is our system matrix. Once system matrix is known, we can correlate the coordinates of image and object planes, this relation we can easily write.

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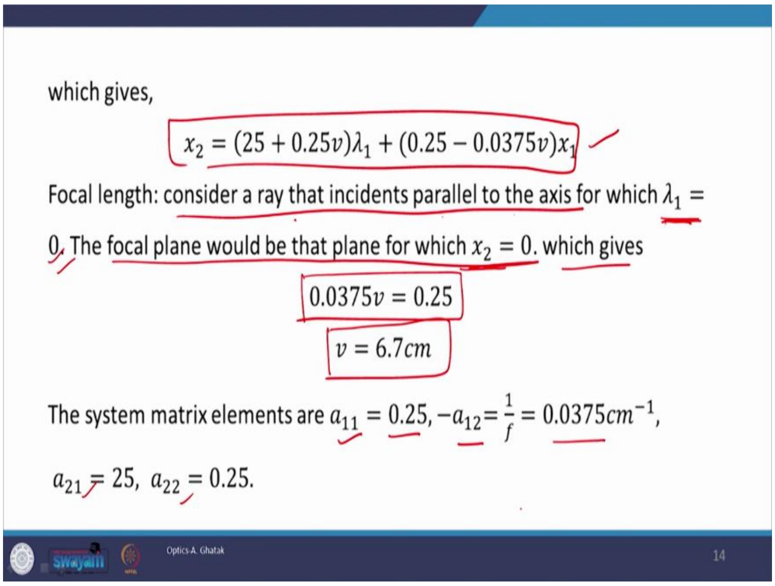

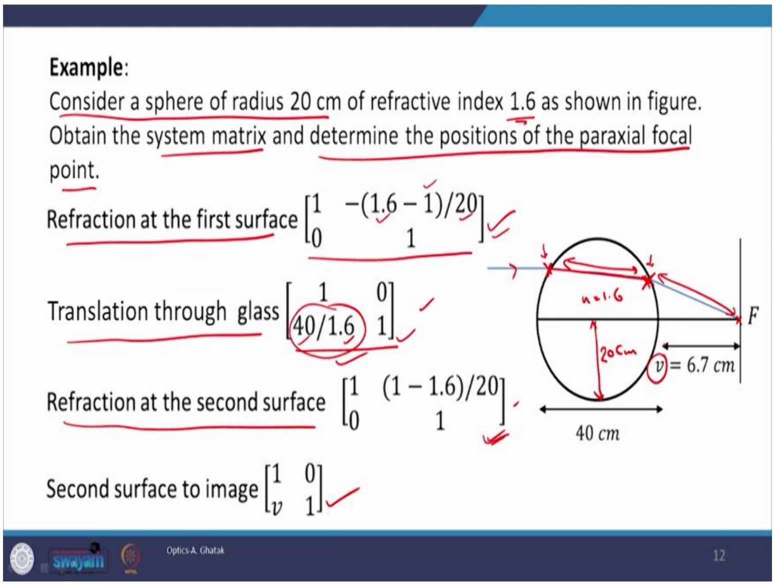

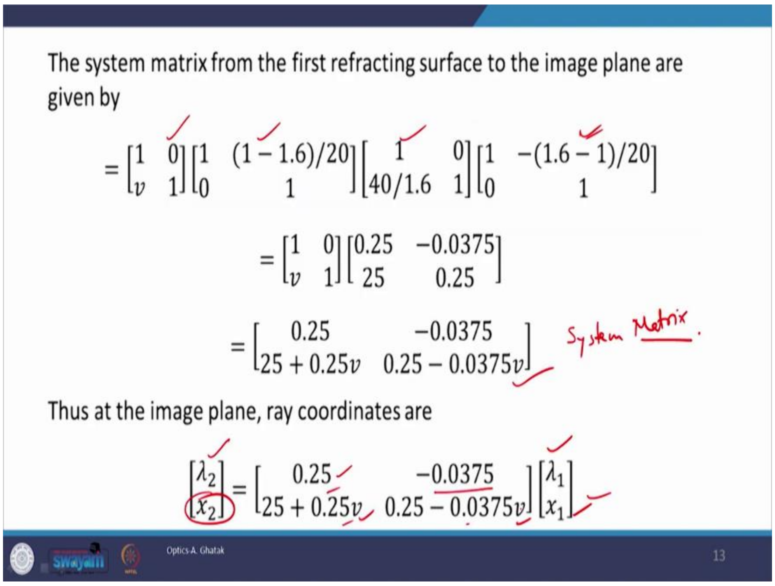

Now, if you expand this matrix to get the expression for x_2 then we will get this expression. Now, the question asks for focal length. Now, consider a ray that incident parallel to the axis, if the ray is coming parallel to the axis then it will not make any angle with the horizontal, what do I mean by saying any angle is that, that the parallel ray will be making zero angle with the horizontal and therefore, its direction cosine $\lambda = n\ sin\alpha$, α is the angle with the horizontal would be equal to 0.

Therefore, λ for the incident ray which is parallel to the axis of the lens system would be equal to 0 and we know from the definition of the focal point is that if a ray is falling parallel to a lens then it will converge to the focal point and what would be the height of the image at the focal point? Since ray is converging to the focal point or we can say that for concave lens the ray will move, after refraction, the ray will follow a path which gives us the feeling that it is emanating or originating from the focal point. The object image would be form at the focal point and the image would be point image.

And since the image is point image, it will not have any height and since it is not having any height x_2 would be 0. Therefore, for ray that is incident parallel to the axis for which λ_1 is equal to 0, the focal plane would be that plane for which x_2 is equal to 0 and which gives if you substitute x_2 and λ_1 is equal to 0 in this first expression. This expression for x_2 then from here you get the expression for v_2 which is 6.7 centimeter. Once the expression for v is known, then you can get all the values in the system matrix.

Then you can get the first element is 0.25, a_{12} will be 1 by focal length, you can calculate it easily. And a_{21} and a_{22} can also be calculated here. This is the elements of the system matrix.

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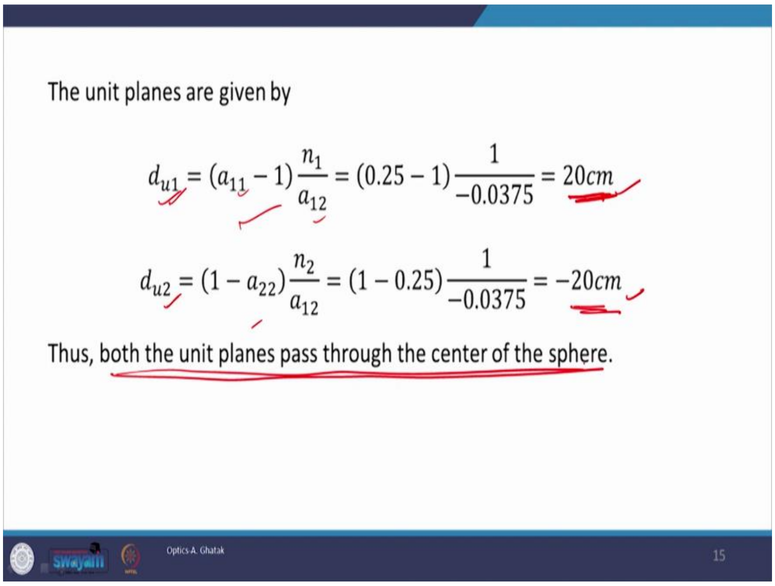

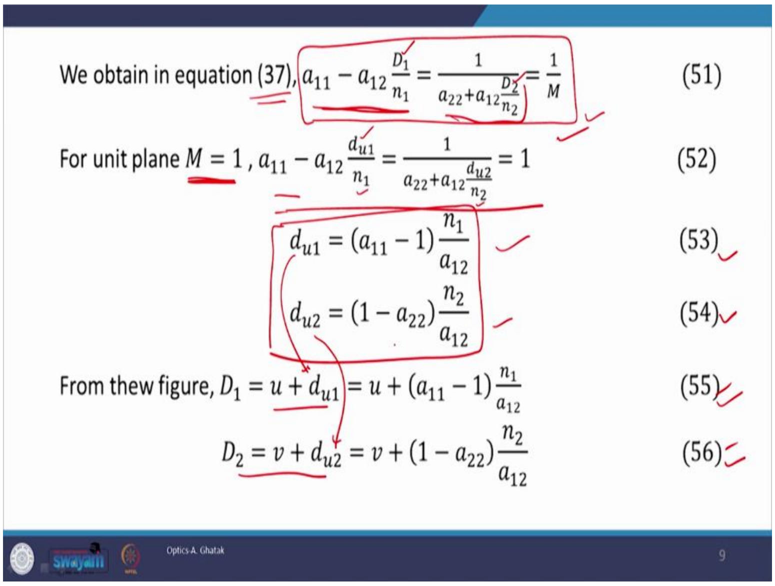

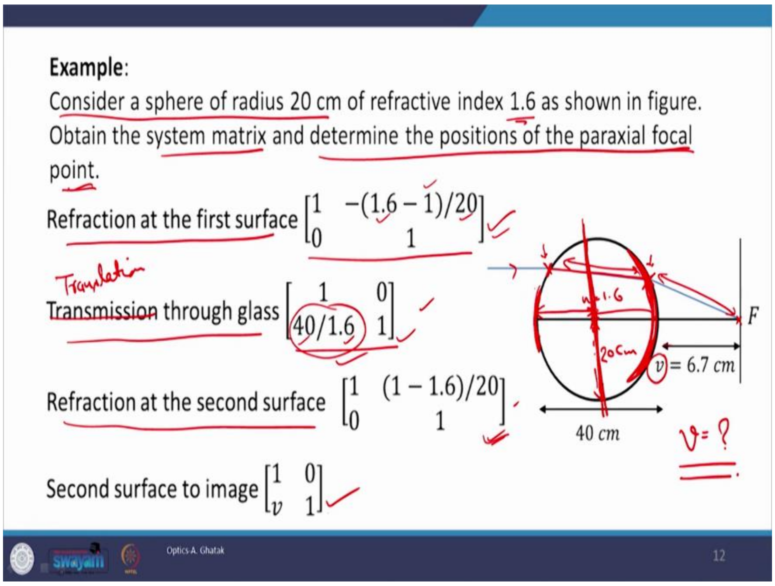

Then once these things are calculated then you can also calculate the positions of unit planes, we know what are d_{u1} and d_{u2}, the expression for d_{u1} and d_{u2} are already derived and these are given here in this previous slides. By equation number 53 and 54 gives the expression for the first and second unit planes, we will use it here. We will use these expressions and substitute the values of $a_{11}, a_{22}, a_{12}, a_{21}$ and n_1 and n_2 and this will give us the positions of unit planes and we get that d_{u1} is equal to 20 centimeter while d_{u2} is equal to minus 20 centimeter.

Now, it means that both the unit plane passed through the center of the sphere, why? because in this sphere, the first sphere, the first unit plane is at 20 centimeter from the left refracting surface and the second unit plane is at minus 20 centimeter from the right refracting surface. So, let us see what is this? This is the left refracting surface and the unit plane is at plus 20 centimeter plus means it will be on the right hand side here. Therefore, first unit plane will pass through the center of the sphere.

Similarly, the second unit plane is at minus 20 centimeter from this refracting surface and the unit plane is at minus 20 centimeter, minus means, it would be on the left hand side of this refracting surface. Therefore, unit plane would be here and 20 centimeter is the radius of this sphere, therefore, this would again be lying at the origin and it would be passing through the origin and the force, both first and second unit sphere would fall on top of each other as it is written here. Both unit planes pass through the center of the sphere and it would be one on top of each other. This is all for today and I will see you in the next class. Thank you for listening me.