Applied Optics
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Lecture: 52
Plane Wave Propagation in Anisotropic Media - I

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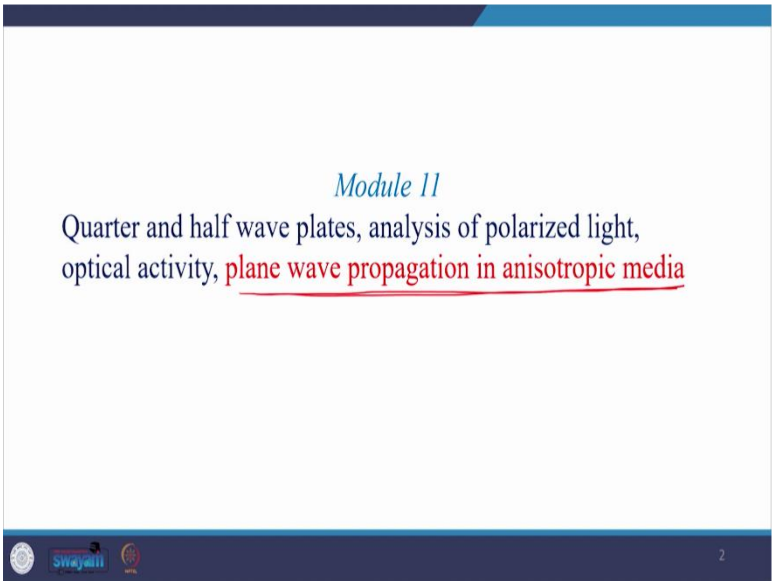

Hello everyone, welcome back to my class. Today, we will study Plane Wave Propagation in an Anisotropic Media. While introducing you double refractive index media or birefringent medium. I introduce n_o and n_e the refractive index for ordinary ray and refractive index for extraordinary ray. Now, the expression for n_o is very simple $n_o = c/v$ but the expression for n_e was quite complicated and it was direction dependent.

Now, after this topic we would be able to see how that expression for velocity of extraordinary ray are refractive index of extraordinary ray was is derived, or what we can say is that the medium properties are direction dependent in an anisotropic medium, while medium properties are direction independent in isotropic medium.

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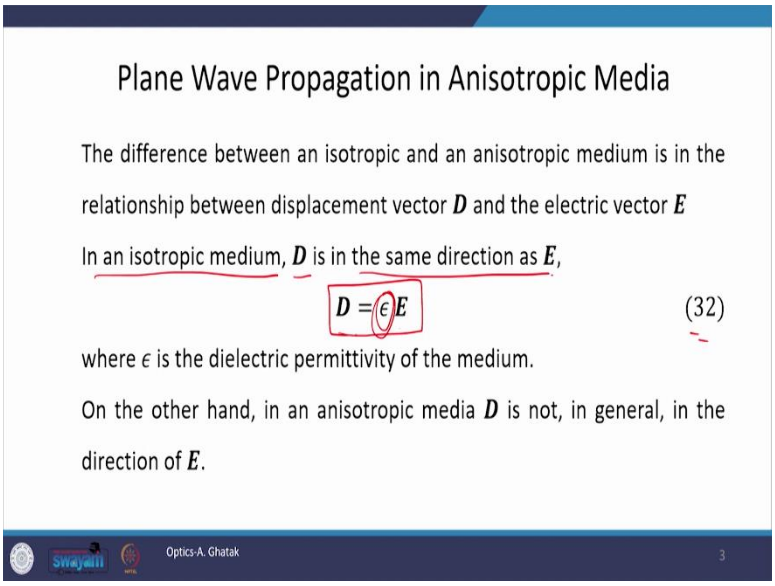

Now, these things are mathematically understood through a relation between displacement vector $\boldsymbol{D}$ an electric field $\boldsymbol{E}$. In anisotropic medium $\boldsymbol{D}$ is in the same direction as $\boldsymbol{E}$ and therefore, the equation number 32 holds for isotropic medium and which equation 32 says that $\boldsymbol{D} = \epsilon \boldsymbol{E}$. Where ϵ is nothing but the usual dielectric permittivity of the medium. On the other hand, in anisotropic medium $\boldsymbol{D}$ is not in general in the direction of $\boldsymbol{E}$, in that particular case the ϵ, the dielectric permittivity it becomes tensor and to represent tensor we resort to matrix representation.

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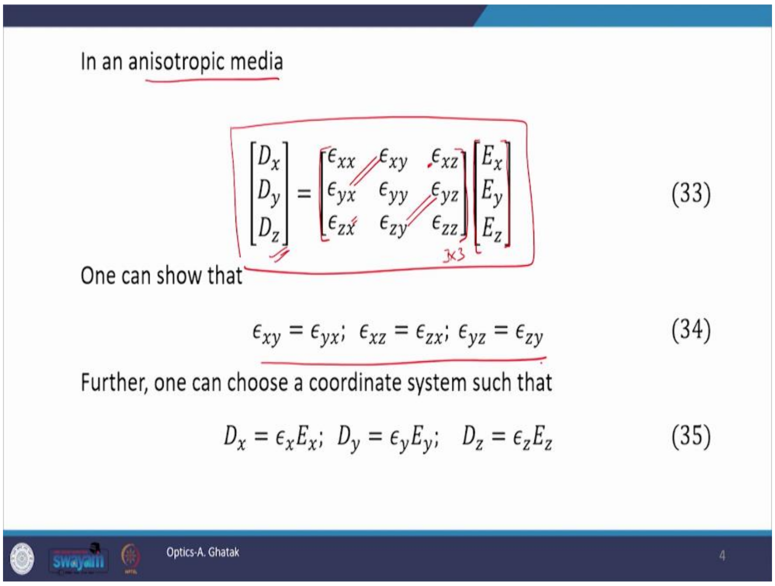

Therefore, in anisotropic medium this relation holds between $\boldsymbol{D}$ and $\boldsymbol{E}$. Where the matrix on the left-hand side is component matrix of displacement vector $\boldsymbol{D}$, this matrix is the component matrix of electric field E and this 3 by 3 matrix, it represents the tensorial nature of the dielectric permittivity.

In usual anisotropic media one can show that off diagonal terms are the same, what I mean to say is that this term is equal to this term while this term is equal to this term well, and this term is equal to this term which are given by equation number 34. With this we can see that we are left with only 6 unknown term in this three by three matrix of permittivity tensor.

Now, if we rotate our coordinate system, then what will happen is that the values of this three by three matrix element changes. Now, if we align the coordinate system in such a way that only diagonal terms are nonzero, then such a coordinate system is called principal coordinate system and the axis are called principal axis. In principal coordinate system or principal axis system, we will be left with only diagonal terms that are ϵ_{xx}, ϵ_{yy} and ϵ_{zz}. Now, for brevity instead of writing ϵ_{xx} from now onwards we will just write ϵ_x.

Therefore, in this rotated coordinate system, we would have these relations $D_x = \epsilon_x E_x$, $D_y = \epsilon_y E_y$, $D_z = \epsilon_z E_z$. Now, this coordinate system is known as as I said before principal axis system and the quantities ϵ_x, ϵ_y and ϵ_z are known as principal dielectric permittivity of the medium.

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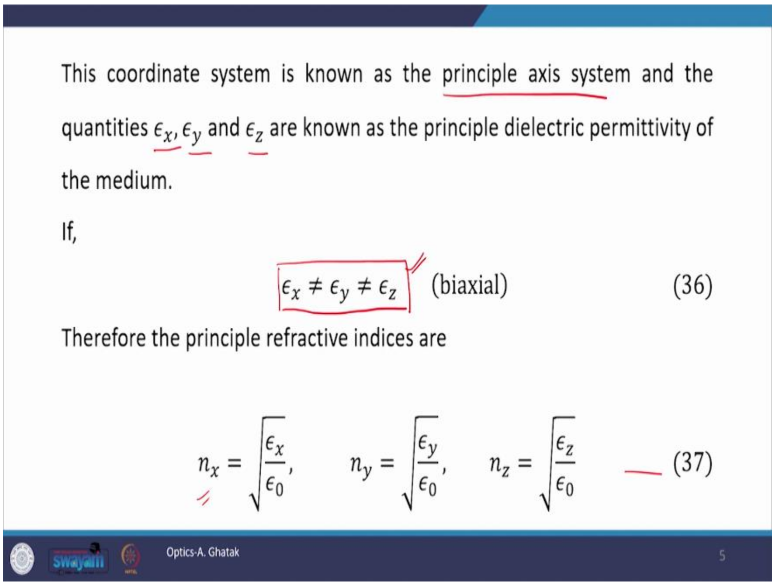

Now, there are two cases, the first case is of biaxial system where in $\epsilon_x \neq \epsilon_y \neq \epsilon_z$. Anisotropic material in which this relation holds is called biaxial material or biaxial crystal. And in these type of crystal the principal refractive indices are given by equation number 37 which says that $n_x = \sqrt{\epsilon_x/\epsilon_0}$. Similarly, $n_y = \sqrt{\epsilon_y/\epsilon_o}$ and $n_z = \sqrt{\epsilon_z/\epsilon_o}$.

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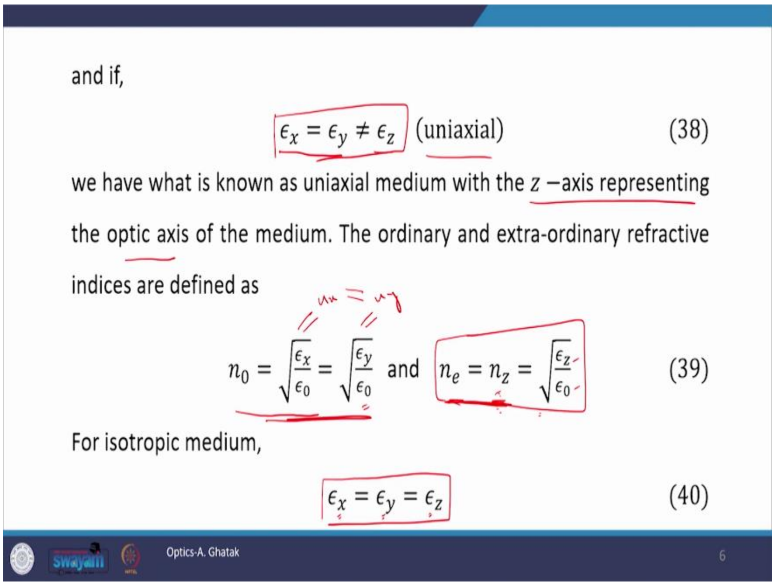

And the second case is when $\epsilon_x = \epsilon_y \neq \epsilon_z$. For the crystals were in this relation holds are called uniaxial crystal. Now, in such a medium z axis represents the optic axis, the ordinary and extraordinary refractive indices in uniaxial crystals are defined as $n_0 = \sqrt{\epsilon_x/\epsilon_0} = \sqrt{\epsilon_y/\epsilon_0}$ and n_e which is extraordinary index which is equal to n_z it would be equal to $\sqrt{\epsilon_z/\epsilon_0}$. Because $n_x = n_y \neq n$.

This is your n_x, this is your n_y, but we know that $n_x = n_y$ because $\epsilon_x = \epsilon_y$ this is not equal to n_z therefore, there is a separate definition n_z or n_e. But for isotropic medium all ϵ are the same, $\epsilon_x = \epsilon_y = \epsilon_z$. Therefore, there is no some specific directions like principal axis.

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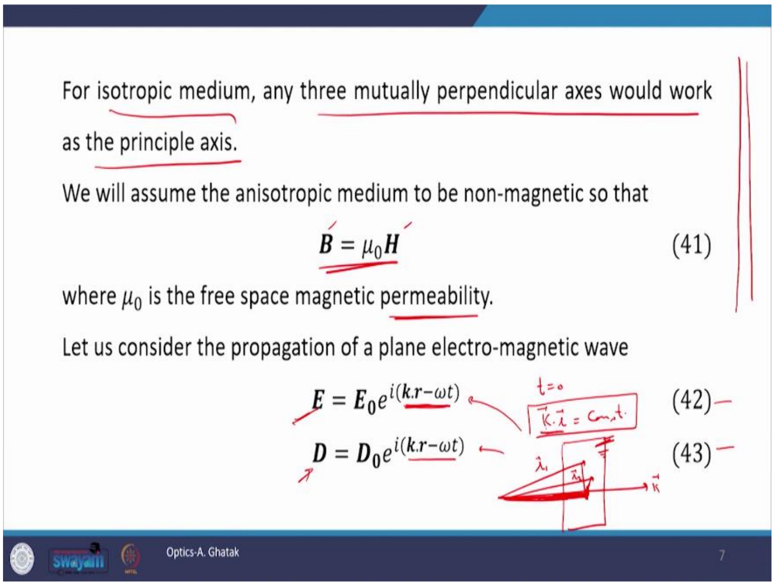

For isotropic medium therefore any three mutually perpendicular axis would work as a principal axis. Now, we will assume the isotropic medium to be nonmagnetic, this would simplify our mathematics therefore, $\boldsymbol{B} = \mu_0 \boldsymbol{H}$. Where μ_0 is the free space magnetic permeability you are hearing these two words permittivity and permeability.

I assume that everybody is aware of **B**, **H**, **D** and **E** but permeability means the ease with which a material be magnetized in an external magnetic field. Similarly, permittivity is the ease with which material be polarized in an externally applied electric field. These were the introduction of the material, there is the properties of the material medium.

Now, we will assume that a plane wave plane electromagnetic wave is propagating in such a material medium. The electric field and the displacement vector for the electromagnetic wave can be expressed by equation number 42 and 43 respectively. Where 42 we have expression for electric field and in 43 we have expression for displacement vector **D**. You see here we have $k.r - \omega t$ in the exponent.

Now, to just let you know that this term $k.r - \omega t$ represents a plane wave because at t=0 we are left with $k.r$ and this equation represents equation of a plane. Because say if this is the direction of k and if you draw a plane here and from this point say this is r_1 and say this is r_2 these are the two vector then the projection of these two vectors on k they will decide k this expression $k.r$.

Now, if r_1 and r_2 are in the same plane, if they are in this plane then only $k.r$ would be constant then only we will get same projection on the k vector direction. If this is the projection, then they will fall on the same point if and only if r_1 and r_2 lies in a plane. This is why this equation 42 and 43 represents plane electromagnetic wave. The phase front is plane, the locus of points which are oscillating in the same phase they create a plane and this is why this wave is called plane wave.

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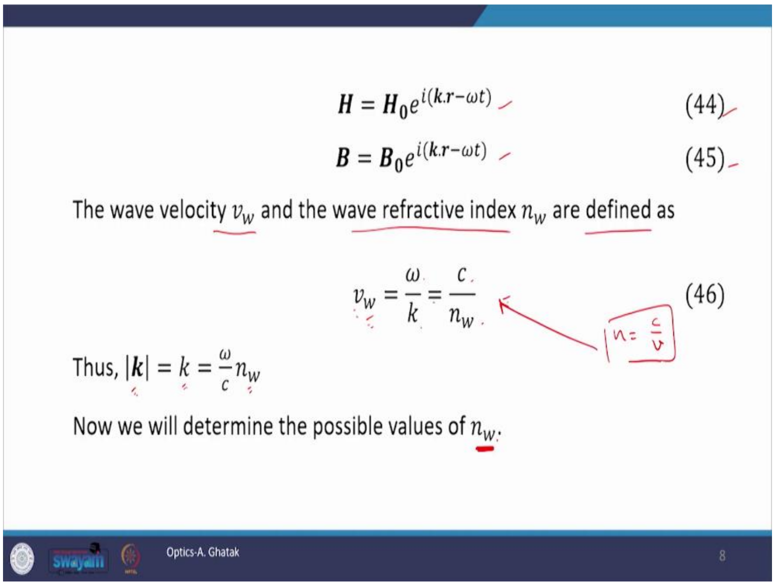

Now, similarly, the expression for H and B field is given by equation number 44 and 45, we know that the wave velocity and the wave refractive index are defined by this relation where $v_w = \omega/k = c/n_w$, this relation we already know. We know that refractive index $n = c/v$ and this is exactly what is written here, instead of writing v and n we are writing n_w and v_w just w represents the wave and wave vector $k = (\omega/c)n$ and if you take the modulus the vector sign with direction will go away and the modulus is equal to $k = (\omega/c)n_w$ we know all these things.

Now, we will determine the possible values of n_w what are the possible values of wave refractive index which we get in birefringent crystal. Now, since the medium is dielectric from the Maxwell equation, first Maxwell equation we get $\nabla . D = 0$. Here we have assumed that the charge density is 0, no charge density.

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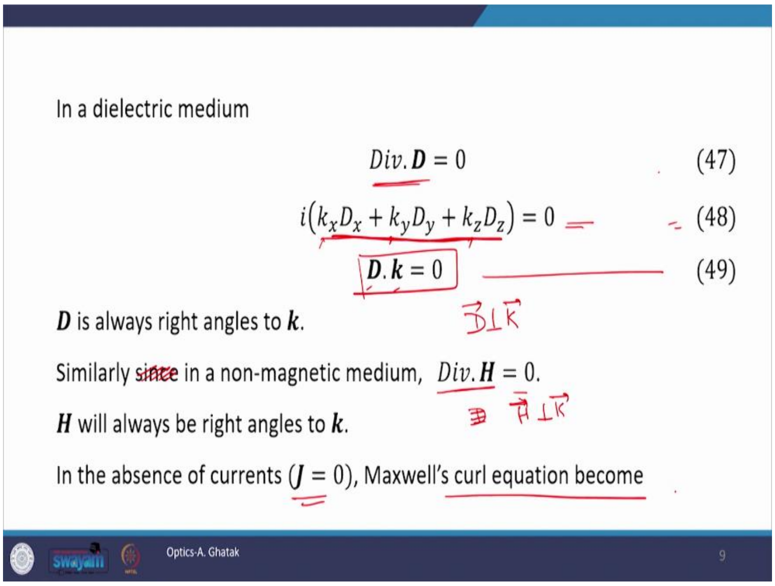

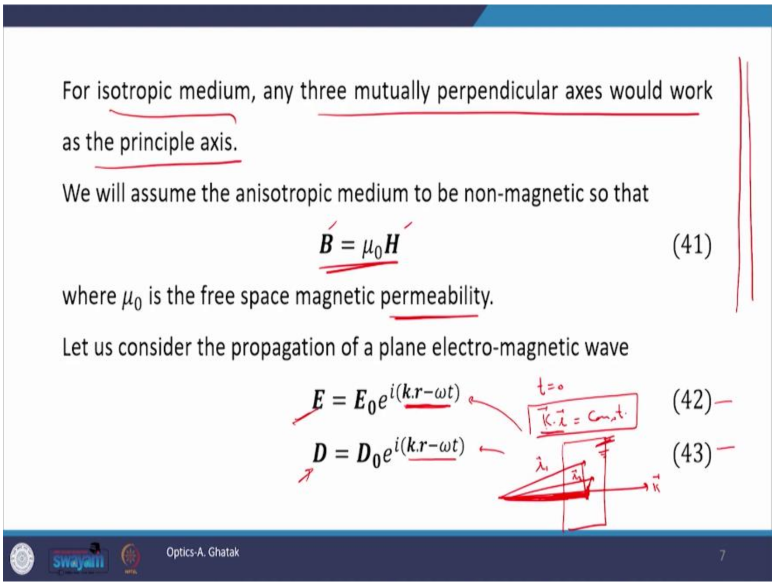

Now, in the component form equation 47 can be written as equation number 48. Now, you see that you are getting here k_y, we are getting k_y because we have used equation this equation for D equation number 43. If you take the divergence of 43 you will get equation number 48. From here you see that this equation is nothing but it is dot product between D and k and this is equal to 0.

Since the dot product between D and k is equal to 0, it means that D is at right angle to k they are perpendicular because the dot product between two quantity A and B is equal to ABcos(θ) and if $\theta = 90$ degrees then this dot product is 0 and therefore, equation 49 says that D is perpendicular to k. Similarly, in a nonmagnetic medium we have divergence of H=0 and from here we can derive H is perpendicular to k.

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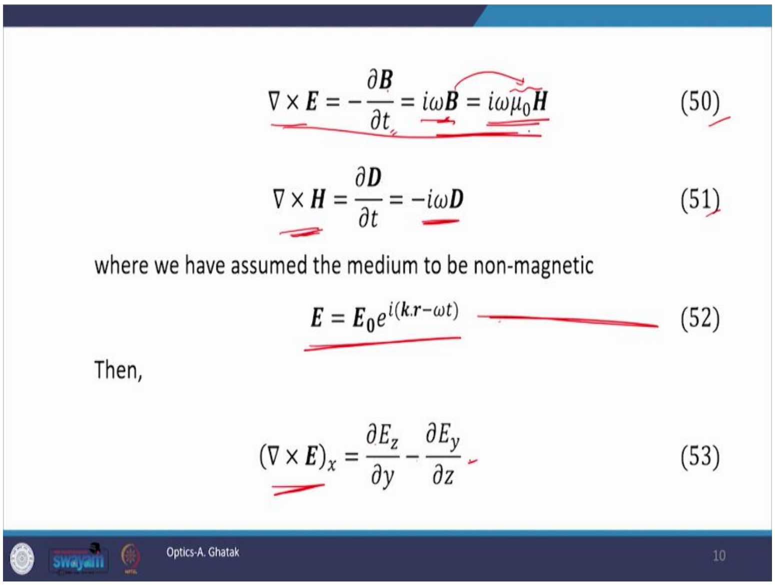

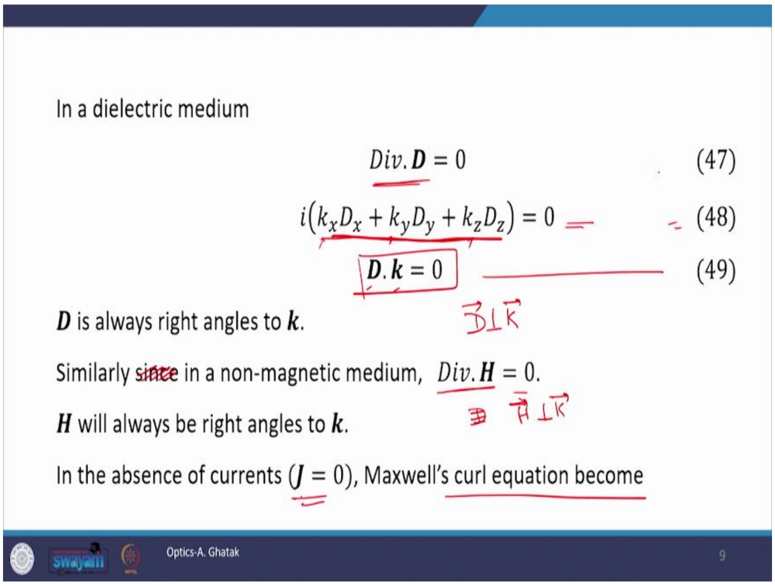

Now, in the absence of current, the Maxwell's curl equation that is curl E and curl H, it can be written in this form equation number 50, $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, and the expression of B we already know it is given here 45. If you take the time derivative of the B you will get $i\omega\boldsymbol{B}$ and $\boldsymbol{B} = \mu_0\boldsymbol{H}$ therefore, you get this final expression for curl H.

Similarly, for $\nabla \times \boldsymbol{H} = \partial\boldsymbol{D}/\partial t + J$ but J=0, we already we have already assumed that there is no current density J is equal to 0 therefore, it is equal to $-i\omega\boldsymbol{D}$. Now, we know that E is expressed as $\boldsymbol{E} = E_0 e^{i(k.r-\omega t)}$. Now, let us take the x component of equation number 50. In light of equation number 52. Now, the x component of curl E would be given by $\partial E_z/\partial y - \partial E_y/\partial z$.

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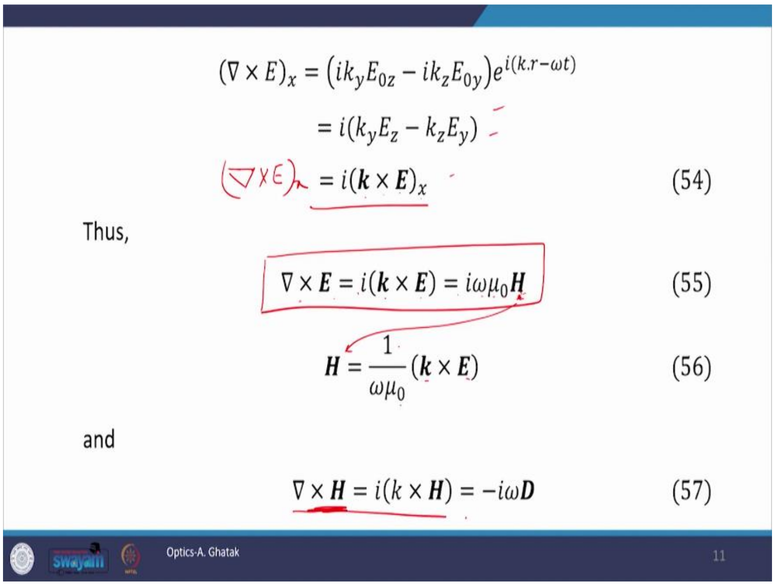

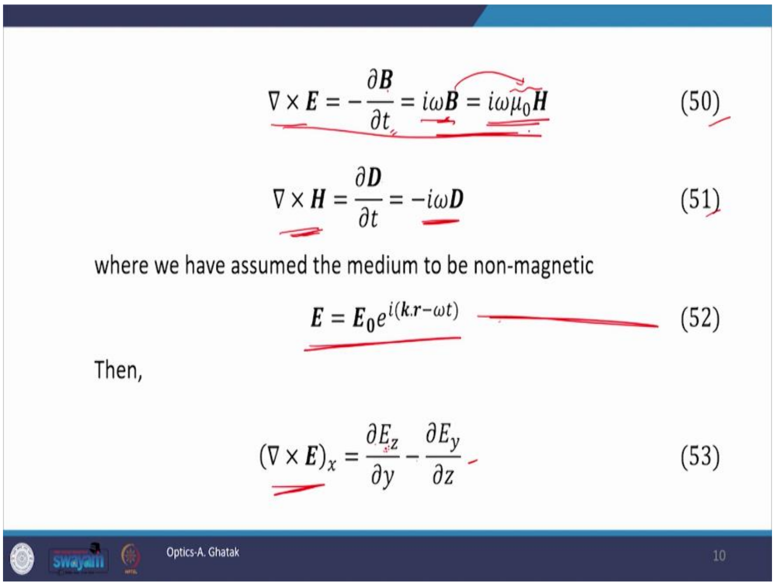

And if you take the derivative of the E_z with respect to y and E_y with respect to z then you get these expressions after a little bit of simplification, the x component of curl E looks like this $i(k \times E)_x$. Therefore, full vectorial equations looks like this if you just combine all x, y and z components then curl E is equal to $ik \times E$ which is equal to $i\omega\mu_0 \times H$. Now, from here you can write the expression of H which is nothing but $1/\omega\mu_0(k \times E)$. Similarly, for curl H the fourth equation like fourth Maxwell equation we can write this expression.

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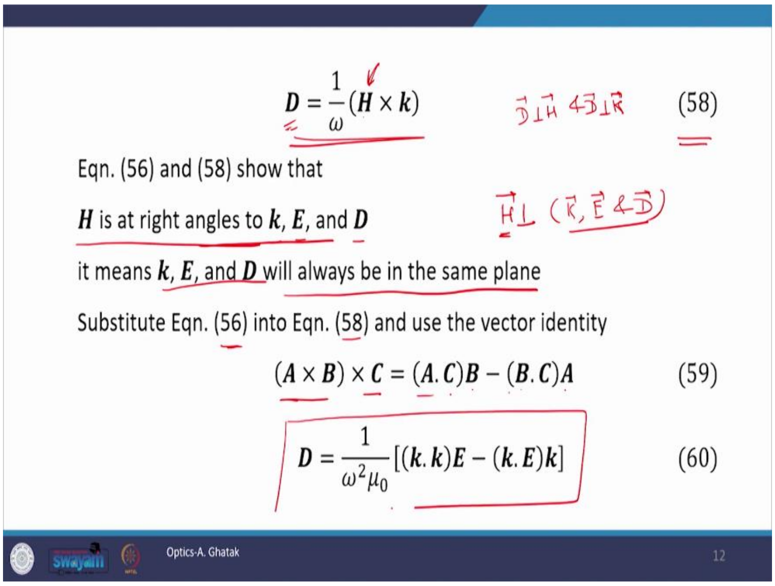

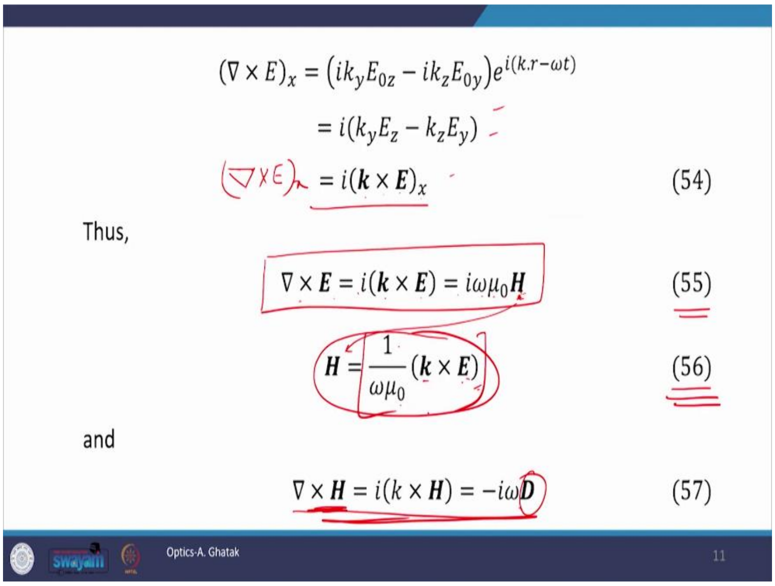

And from here we get the expression for D here D is here on the right-hand side and the expression $\boldsymbol{D} = 1/\omega(\boldsymbol{H} \times \boldsymbol{k})$. It means D is perpendicular to H as well as D is perpendicular to k. This is what equation 58 says. Similarly, this equation says that H is perpendicular to k as well as H is perpendicular to E.

Now, it means H is at right angle to k, E and D, these equations basically with equation number 56 and equation number 58 we conclude that H is at right angle to k, E and D. And it means k, E and D will always be in the same plane, which is very much obvious, H is not perpendicular to three quantities. Therefore, k, E and D will always be in the same plane.

Now, we will substitute equation number 56 into 58 here this is our equation number 56 which has the expression of H we will substitute it into 58. We will replace this H with the right-hand side of equation number 56 this. With this substitution and we will use this identity which says that $A \times B \times C = (A.C)B - (B.C)A$ with this we get this expression for displacement vector D.

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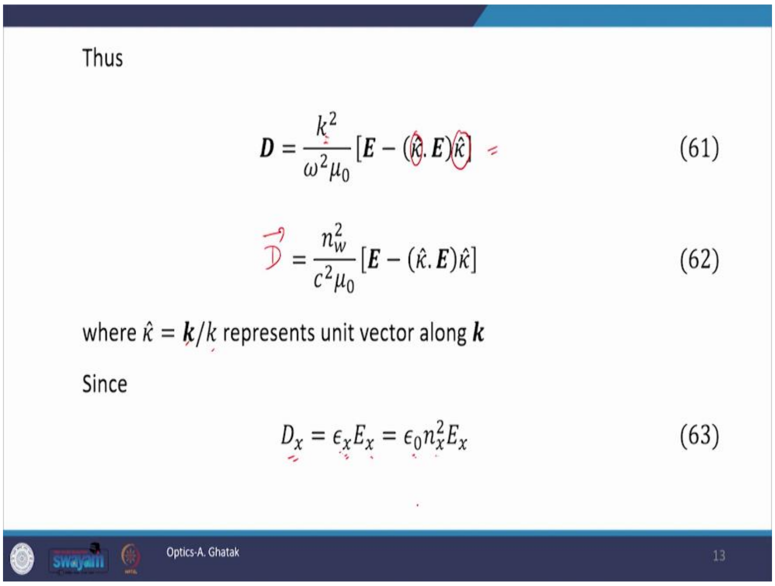

Now, let us simplify it we take k the magnitude of k outside of this bracket then equation number 60 becomes a equation number 61. A further simplification leads to equation number 62 you are seeing $\hat{k}.\ \hat{k} = \vec{k}/|\vec{k}|$ which is unit vector which points along vector k, and we know that $D_x = \epsilon_x E_x$ which is in principle coordinate system and ϵ_x is nothing but $\epsilon_0 n_x^2$ which we already which we have already seen.

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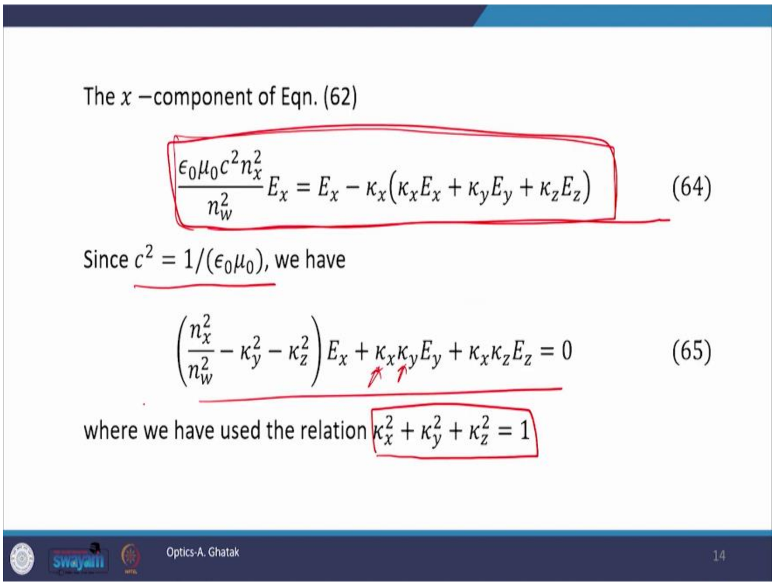

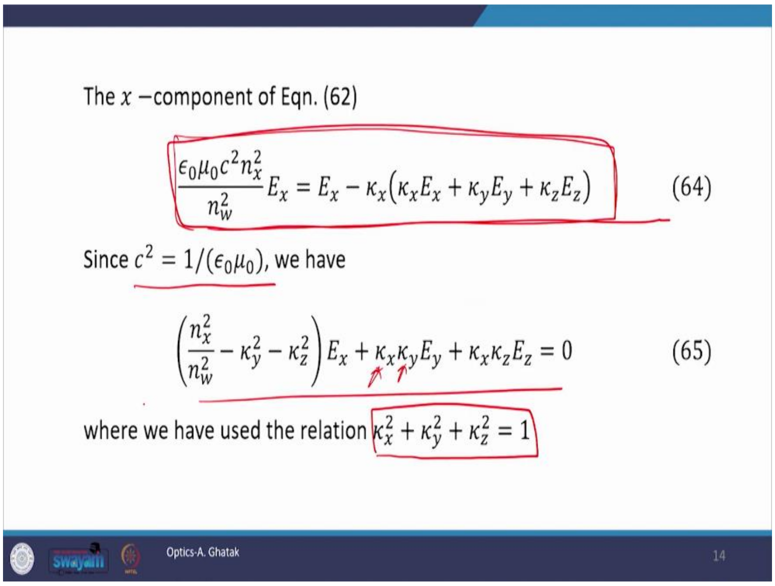

Therefore, the x component of equation number 62 can be written as this. The equation number 62 the x component of equation number 62 is written like this here. Where in what we did is that we pick the x component of D then the x component of E and x component of k, $\hat{k}$ which is outside the bracket. Now, we already know that $c^2 = 1/\epsilon_0\mu_0$ bit of simplification gives equation number 65.

This looks a bit complicated and here we have used that $k_x^2 + k_y^2 + k_z^2 = 1$, this is property which you can use easily here. Now, do notice that this k which is different from wave vector k or unit vector, these are the components of the unit vector $\hat{k}$ in this let us call it κ, κ_x κ_y they are and $\kappa_z$they are component of the unit vector $\hat{k}$.

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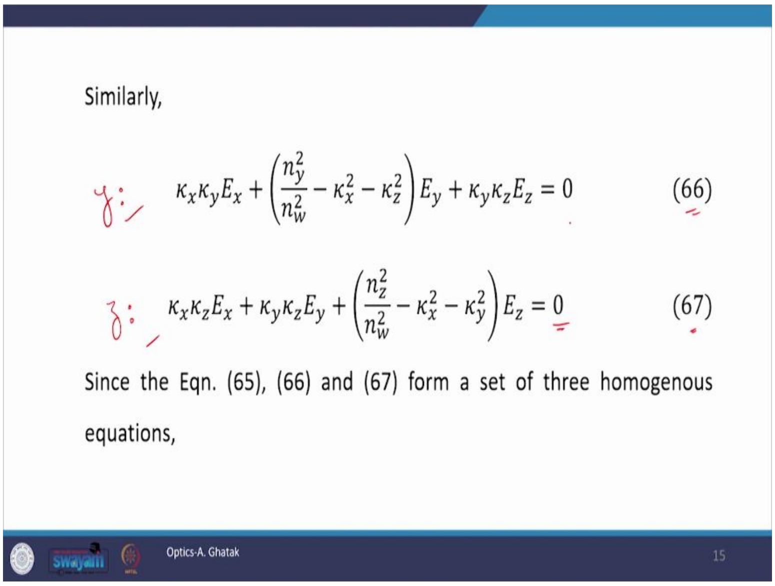

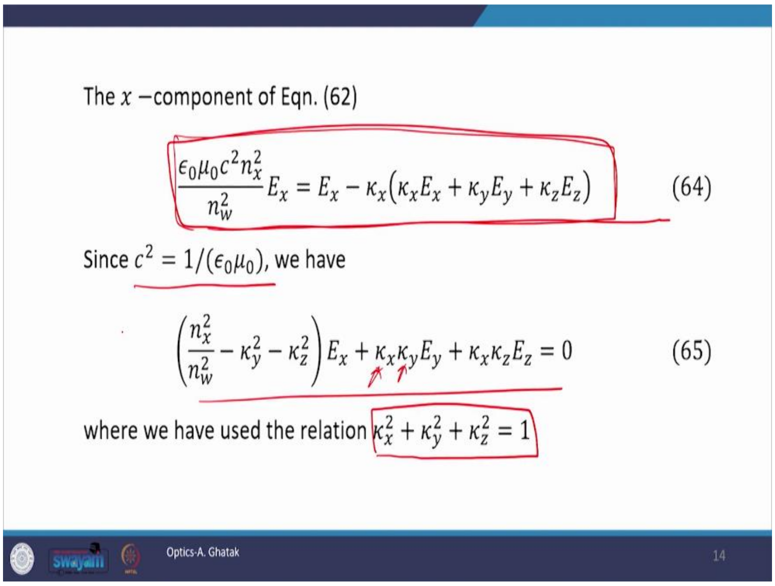

Now, similarly, we can write the y component as well as z component. This is our E_x component this is y component and this is z component of the same equation. Now, these equations equation number 65, 66 and 67 they are the component of the same equation and there are three homogeneous equations you see on the right-hand side here we have 0.

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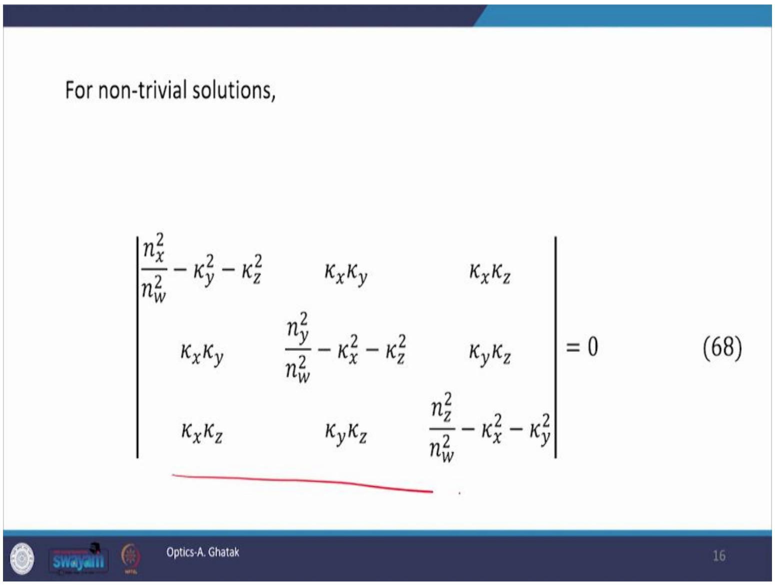

Therefore, for non-trivial solutions the determinant of the coefficient matrix must be 0. Therefore, we equate this coefficient matrix to 0.

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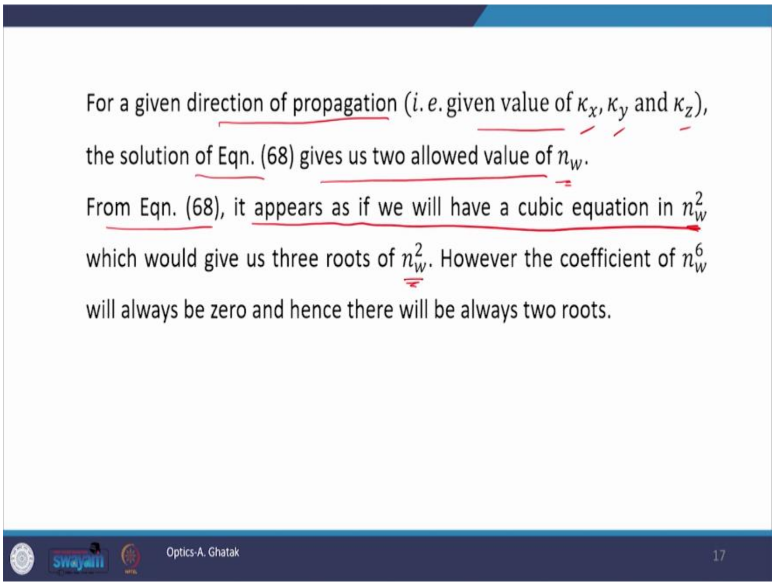

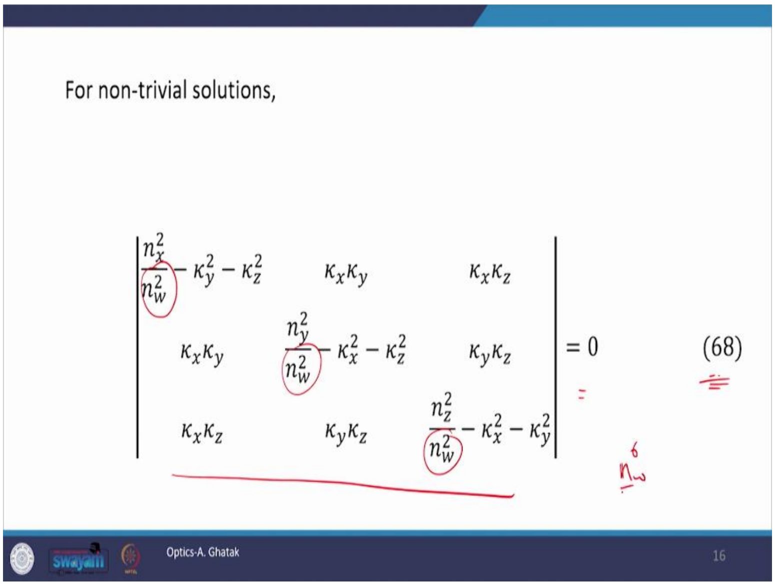

Now, for a given direction of propagation that is for a given value of κ_x, κ_y and κ_z, the solution of this equation number 68 this determinant gives us two allowed values of n_w. But if you look into equation number 68 then you say that n_w^2 is coming here, here, here. It means that we will have an equation which will have n_w to the power 6 as dominant order and then the order will appear in this equation.

Now, therefore, from equation 66 it appears as if we will have a cubic equation in n_w^2, it means the dominant order would be n_w^2 to the power 3 that is n_w^6. Which would give us three roots of n_w^2, but the coefficient of n_w^6 will always be 0 and hence there will be always two roots, there would be only two roots of n_w^2. Now, I stopped today's lecture here, we will continue with this derivation in the next lecture and then we will see what would be the final expression of n_e and n_0 in uniaxial crystal. Thank you for joining. See you in the next class.