Applied Optics
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Lecture 50
Quarter and Half Wave Plates

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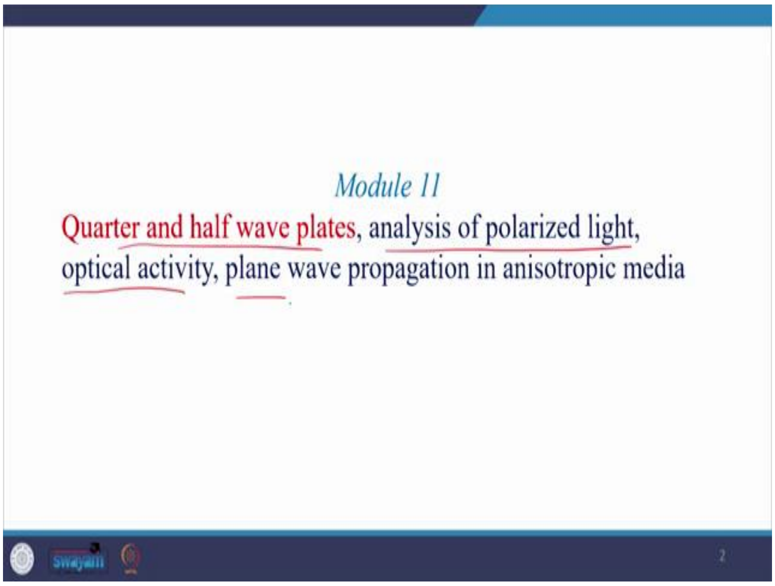

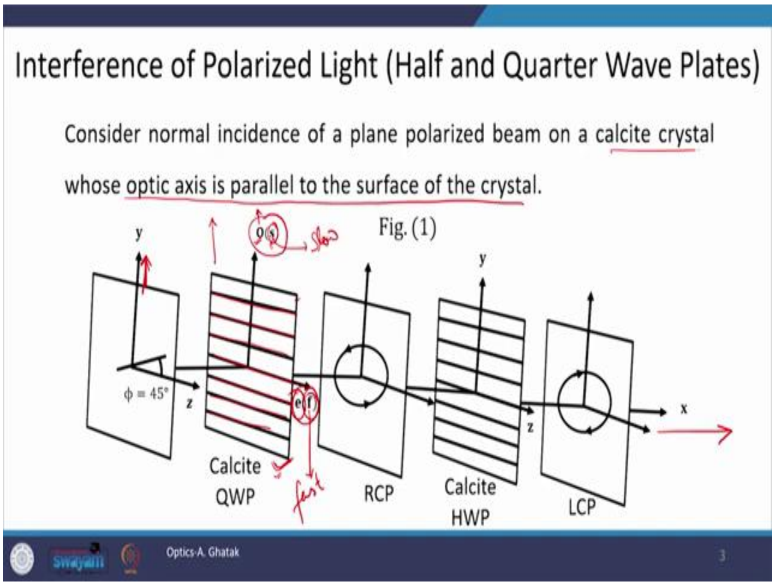

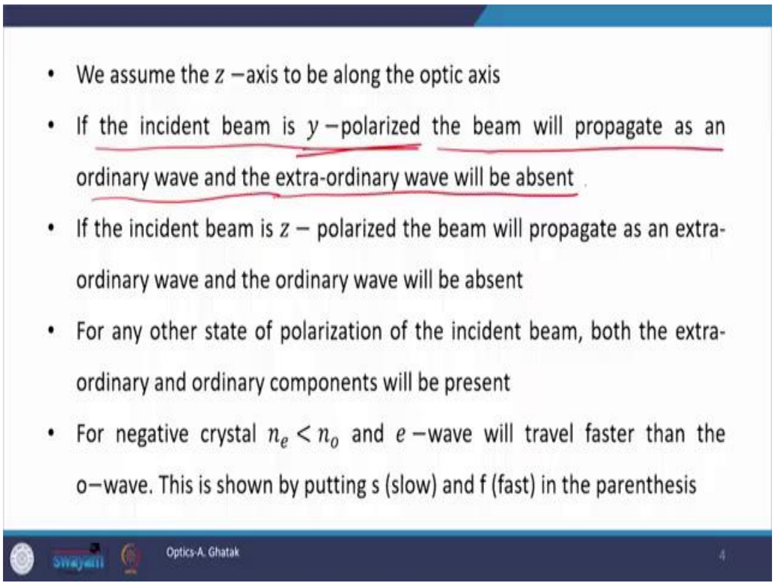

Hello everyone, welcome back to my class, today we will start new module which is module number 11. In this module we will talk about quarter and half wave plate and then we will learn the ways of analyzing polarized light. Then the concept of optical activity will be introduced to you and at last theoretical foundations to study plane wave propagation in anisotropic media would be talked about.

Now, we will today start with this quarter and half wave plate, which is a prominent and significant application of what we learned in our previous module which was majorly devoted to double reflection. Now, the half and quarter waves are nothing but they are interference of polarized light, but what type of interference? We know that double refractive medium it splits unpolarized light into two parts, first is called o ray and the second is called e ray.

Now, if we somehow make these rays interfere then we will get or we will see different kind of control over the light. Now, to understand the concept of half and quarter wave plate let us consider the normal incidence of a plane polarized beam on a calcite crystal, keep it in mind that calcite crystal is a negative crystal.

Now, in this calcite crystal, we assume that optic axis is parallel to the surface of the crystal and this crystal is shown here in this figure number one and you see that the optic axis is parallel to the surface and it is pointing along z axis. z axis is coming out of the paper, in the right-hand direction this is the direction of wave propagation which is x axis, the vertical axis is y axis, while the axis which is coming out of the paper it is z axis and along the optic axis of this calcite crystal is along this z axis.

Now, if we launch a beam which is y polarized then let us see what happens, now if a beam is y polarized that is if the polarization is vertical then you see that this polarization is perpendicular to the direction of propagation which is $\vec{k}$ the wave vector and this polarization is also perpendicular to the optic axis. It means y polarized light will behave as o beam or o ray for the calcite crystal which is oriented like this.

Therefore, if the incident beam is y polarized, the beam will propagate as an ordinary wave and the extraordinary wave will be absent. Similarly, if the incident beam is z polarized then from the figure you can see that the direction of polarization now will be along the optic axis while it would be perpendicular to the $\vec{k}$ that is direction of propagation x axis and this polarization is also in a plane which is formed by optic axis and the $\vec{k}$ vector.

Therefore, a z polarized beam will behave as e wave in calcite crystal and therefore there will not be any o wave or ordinary wave. It is all the game of polarization now, but for any other state of polarization of the incident beam both extraordinary and ordinary components will be present.

First, we talked about two extreme cases where in only one type of wave is present either ordinary or extraordinary, but now if the state of polarization is such that the orientation of the vibration is neither along z nor along y, but it is along at angle with respect to z then the both ordinary and extraordinary component will be present.

Now, for negative crystal like calcite where n_o is larger than n_e, the extraordinary wave will travel faster than the ordinary wave because n_e is smaller therefore v_e would be larger the velocity of extraordinary wave will be larger therefore extra ordinary wave will travel faster, therefore we represent the polarization which only support extraordinary ray as fast direction and the polarization direction which only supports ordinary ray it is represented as slow direction.

And which is shown here, you see this is in these two direction, you see that there is o and in parenthesis s, o stand for ordinary wave and s is slow, similarly e stands for extraordinary wave and f means fast, this is fast direction while this is slow direction. Why, because calcite crystal is negative crystal and extraordinary ray travel faster and if a ray is polarized along z direction it will only support e wave and therefore the e word is here, similarly for o also.

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Let the electric field vector (of amplitude E_0) associated with the incident polarized beam make an angle ϕ with the z −axis.

Such a beam can be assumed to be a superposition of two linearly polarized in phase beams polarized along the y −and z −directions with amplitudes $E_0 \sin\phi$ and $E_0 \cos\phi$ respectively.

- The z −component passes through as an extra-ordinary beam propagating with wave velocity (c/n_e)
- The y −component passes through as an ordinary beam propagating with wave velocity (c/n_0)

Since $n_e \neq n_o$, the two beams will propagating with different velocities and when they came out from the crystal, they will not be in phase.

The y − and z −components of the incident beam can be written in the form

$$E_y = \boxed{E_0 \sin\phi} \cos(kx - \omega t) \quad (1)$$

$$E_z = \boxed{E_0 \cos\phi} \cos(kx - \omega t) \quad (2)$$

Let $x = 0$, represents the surface of the crystal, then at $x = 0$,

$$E_y(x = 0) = E_0 \sin\phi \cos(\omega t) \quad (3)$$

$$E_z(x = 0) = E_0 \cos\phi \cos(\omega t) \quad (4)$$

Inside the crystal, the two components will be given by $k x n$

$$E_y = E_0 \sin\phi \cos(n_0 k x - \omega t) \quad (5)$$

$$E_z = E_0 \cos\phi \cos(n_e k x - \omega t) \quad (6)$$

If the thickness of the crystal is d, then at the emerging surface, we will have

$$E_y = E_0 \sin\phi \cos(\omega t - \theta_0) \quad (7)$$

$$E_z = E_0 \cos\phi \cos(\omega t - \theta_e) \quad (8)$$

where,

$\theta_0 = n_0 kd$ and $\theta_e = n_e kd$

Now, let the electric field say of amplitude E_0 is incident and it is polarized in such a way that it makes an angle φ with the z axis, which is shown here in this figure. The angle φ is here and this angle is made with respect to z axis and the amplitude of the incident wave is E_0, and since this linearly polarized incident wave is polarized at angle φ with respect to z axis then such a beam can be assumed to be a superposition of two linearly polarized in phase beam that are polarized along y and z directions.

We can always split this linearly polarized beam in its z and y components, because it is making an angle phi with a z component and therefore with the y axis it will make $90 - \varphi$ degree angle and you can always split it into two components. Since the incident amplitude is E_0, the amplitudes after this decomposition along y and z direction would be represented by $E_0 sin\theta$ and $E_0 cos\theta$ because this is z axis this is our y axis and this is the polarization direction therefore along z we will have $E_0 cos\varphi$ while along y we will have $E_0 sin\varphi$ components.

The z component which is along optic axis, it passes through as an extraordinary beam and since it is passing through an extraordinary beam it will have a velocity which is equal to c/n_e while the y component it passes through as an ordinary beam propagating with a velocity c/n_o or n_0. Now, we know that n_o is not equal to n_e, therefore the two beams they will come out of the crystal with different phase, since the velocities are not same therefore they will come out with different phase.

The y and z component of this incident beam can be written in this form which is given by equation number 1 and 2 respectively, the y component is given by $E_0 sin\varphi cos(kx - \omega t)$ while z component is given by $E_0 cos\varphi cos(kx - \omega t)$, x is the direction of the propagation and these are the $E_0 sin\varphi$ and $E_0 cos\varphi$ is the amplitude and this is the incident beam, and it is propagating in the direction x.

Now, at the first surface let us associate the coordinate system in such a way that $x = 0$. We have this calcite crystal and at the first surface of the calcite crystal we assume that $x = 0$, in this case at the input phase of the calcite crystal the y and z components are represented respectively by equation number 3 and 4, wherein we just substituted x with zero. Now, at the input we know the form of the field let us calculate the field at the output.

Now, say that crystal has certain thickness but the wave propagate through this thickness that at any depth inside the crystal the y component of the field can be written as

$E_0 sin\varphi cos(n_0 kx - \omega t)$ where x is the depth inside the crystal, n_0 is its refractive index and k is the wave vector.

Similarly for z component which is propagating as extraordinary wave, the field would be written as $E_0 cos\varphi cos(n_e kx - \omega t)$, earlier at the input phase of the crystal when x was zero it was only $cos\omega t$ in both the cases, in both the component but as soon as it enters into the crystal, the x would be non zero and we will have $n_e kx$, had it been a vacuum it would have been kx only. But since the waves are traveling in a medium we will have to multiply kx with refractive index as we have done in earlier cases in geometrical optics.

Now, the phase contribution would be $kx \times n$, but we have two types of wave ordinary wave and extraordinary wave therefore we will have two n's; for ordinary wave we will write n as n_0 while for extraordinary wave we will write n as n_e and this is why we have equation number 5 and 6.

Now, if the crystal thickness is d then at the other phase of the crystal at the emerging surface of the crystal we will have the fields which would be given by equation number 7 and 8 respectively. Here the θ_0 and θ_e are given by these expressions. What we have done here is that we just replace x by d because the total thickness of the crystal is d, therefore $n_0 kd$ is here in the equation number 7 and $n_e kd$ is there in equation number 8 and they are expressed by θ_e and θ_e respectively.

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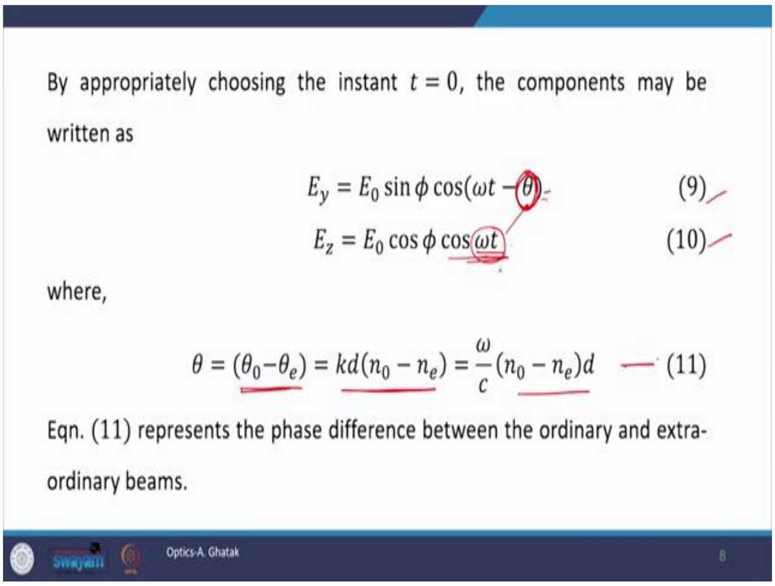

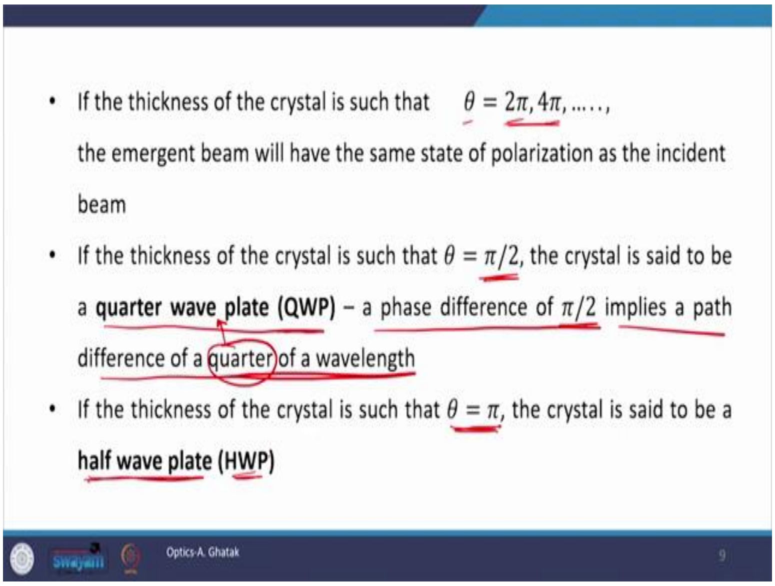

Now, it is the relative phase between the waves which ultimately matters, therefore we can tune our instant $t = 0$ in such a way that the components E_y and E_Zs are written like this here. We shifted all the phase from equation number 10 to equation number 9 because ultimately it is relative phase between the waves which matters.

Phase is not an absolute quantity, it is always measured with respect to something, with reference to something we made E_Z as our reference therefore in the phase we have ωt only left here in equation number 10, and all the relative phase appear in equation number 9 only where θ is nothing but $\theta_o - \theta_e$.

Earlier θ_o was there in equation number 7, θ_e is in equation number 8, now θ is only appearing, the relative phases only appear in equation number 9 where $\theta = \theta_o - \theta_e$. If you substitute the expressions of θ_o and θ_e then you get this form which is given by equation number 11. Equation number 11 is nothing but phase difference between the ordinary and extraordinary beam.

Now, if the thickness of the crystal is such that $\theta = 2\pi, 4\pi$, then from equation number 9 and 10 you can see that the E_y and E_z components are the same, they do not have any difference in their expression, under this situation the emergent beam will have the same state of polarization as the incident beam and the input, the E_y and E_z component, were same but due to the different refractive indices for o and e ray, the crystal introduced different phases to these two rays and due to these two differences we were expecting that the state of polarization will change but if phase introduced by the crystal is integral multiple of 2π then the state of polarization of the incident beam will be the same.

Now, if the thickness of the crystal is such that the relative phase difference is $\pi/2$ then the crystal is said to be a quarter wave plates, QWP, because $\pi/2$ is one quarter of 2π, the whole circle. A phase difference of $\pi/2$ implies a part difference of quarter of a wavelength and this quarter is borrowed in quarter wave plate.

Similarly, if the thickness of the crystal is such that θ, the phase difference, introduced is π then the crystal is said to be half wave plate again because π is one half of full circle, 2π, this is equivalent to path difference of half of a wavelength therefore a crystal which introduces a phase difference of π is called half wave plate or the acronyms for half wave plate is HWP.

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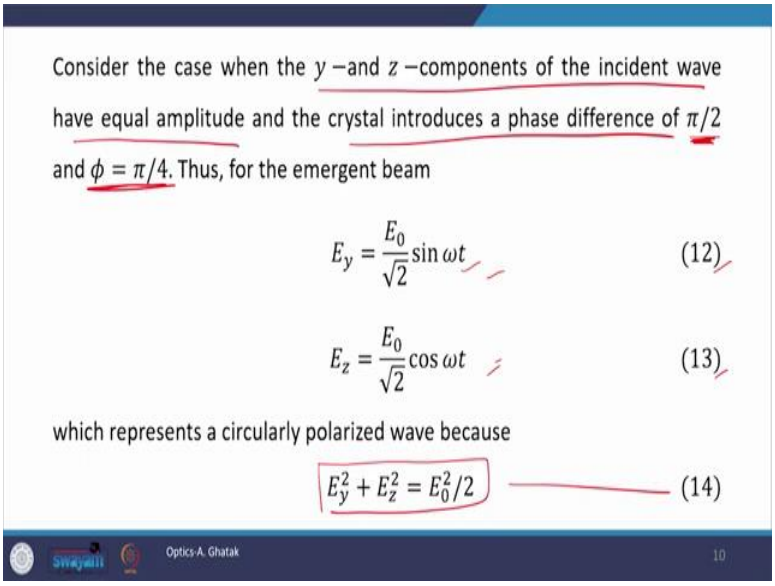

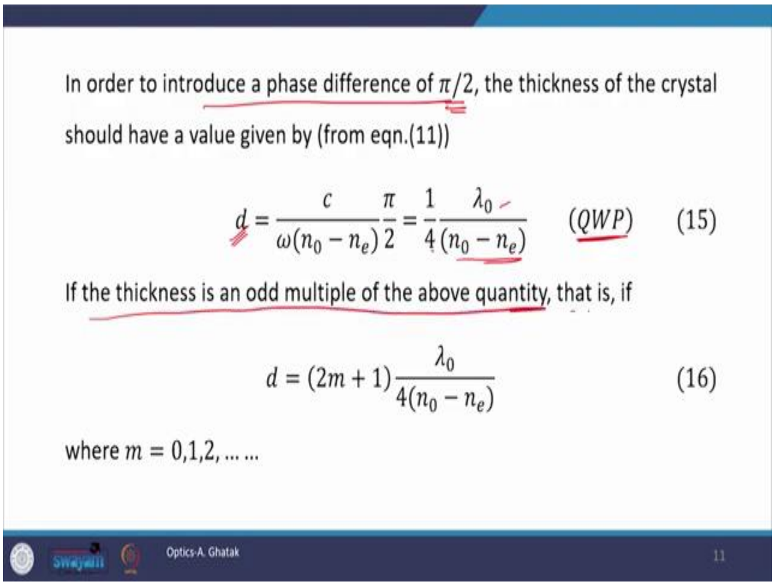

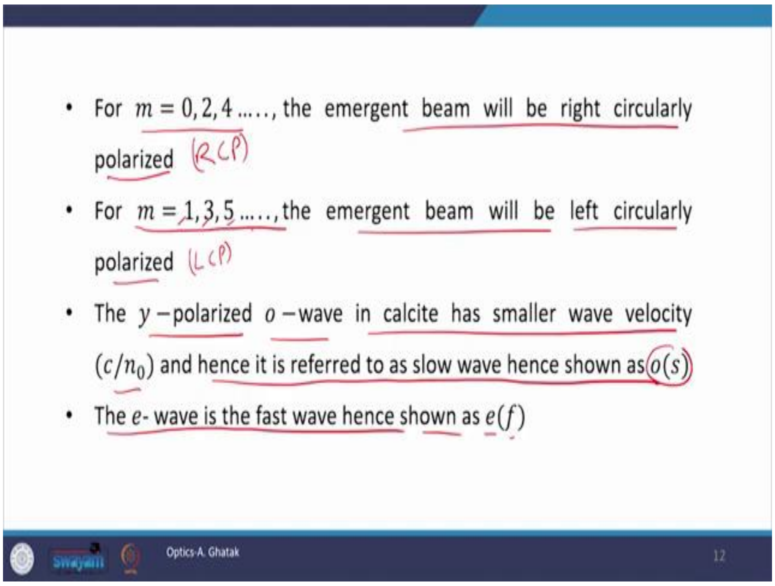

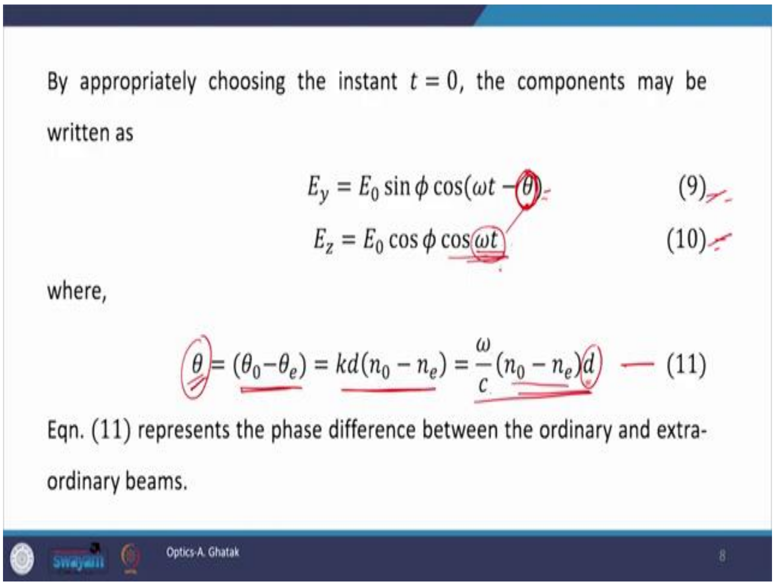

Now, as a example let us consider a case when y and z component of the incident wave has equal amplitude and the crystal introduces a phase difference of $\pi/2$, half wave plate and φ, the initial state of polarization is such that $\varphi = \pi/4$. If you substitute for θ and φ in expression number 9 and 10 then we get equation number 12 and 13, and from equation 12 or 13 you can readily check that these are nothing but they combinedly represent a circularly polarized light. If you square and add them you get equation of a circle as is given by equation number 14.

Then what we found is that if we launch a light which is polarized at angle 45 degree with respect to the z axis and if this light is incident on a quarter wave plate then this linearly polarized light is converted to a circularly polarized light and this is the beauty of these wave plates, they convert the polarization.

Now, in order to introduce a phase difference of $\pi/2$, we will have to tune the thickness as per equation number 11. Let us go to equation number 11 and check now the phase difference theta this you can see is equal to $\omega/c\ (n_0 - n_e)d$. We want $\theta = \pi/2$ for this the only variable is d, now if we tune d such that $\theta = \pi/2$ then we will have a wave plate which is called quarter wave plate.

Therefore, if you substitute $\theta = \pi/2$ then you will get an expression for $d = \lambda_0/4(n_0 - n_e)$ and if we have a crystal such a thickness then this crystal is called quarter wave plate. In additions, if the thickness is an odd multiple of the above quantity that is if $d = (2m + 1)\lambda_0/4(n_0 - n_e)$.

In this case if m=0,2,4,… then the emergent wave will be right circularly polarized, RCP, for $\theta = \pi$, $\varphi = \pi/4$ and if m=1,3,5… and so on then the emergent wave will be left circularly polarized, LCP. Always remember that y polarized o wave in calcite has smaller velocity and hence it is referred as slow wave, hence shown as o in bracket s and e wave is the fast wave hence shown as e in bracket f, this is are about quarter wave plate.

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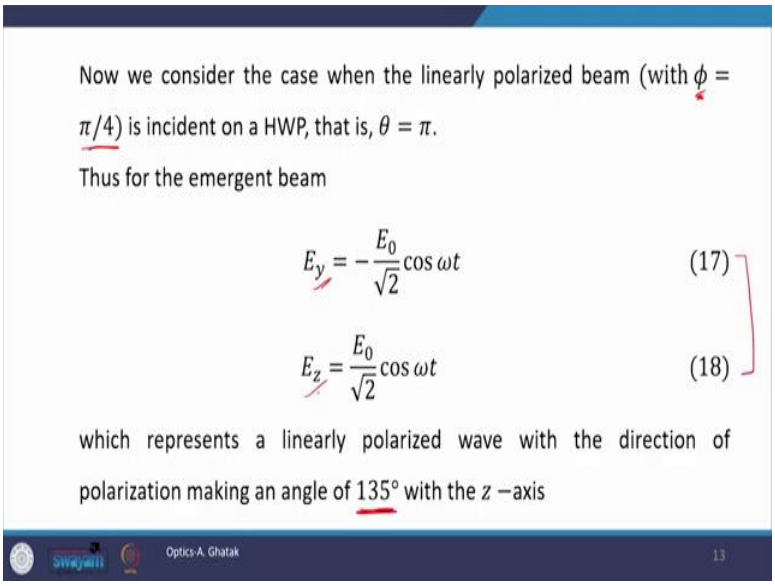

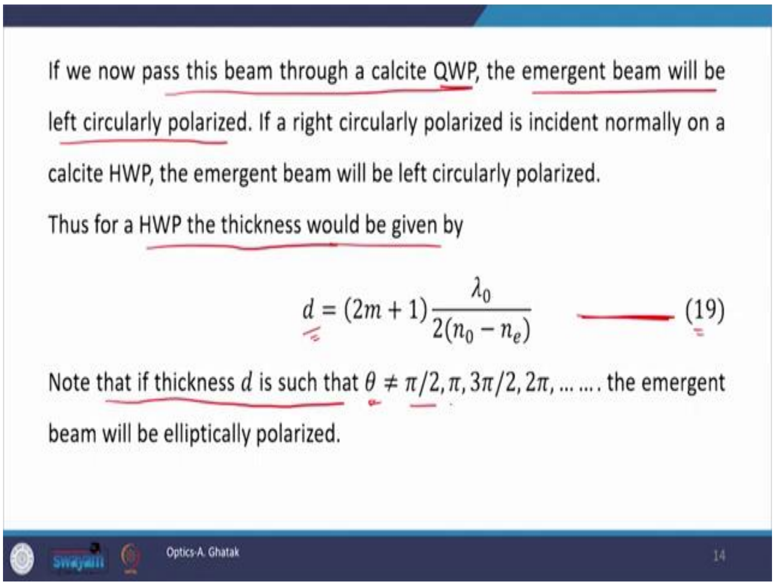

Now, we will see what will happen if $\theta = \pi$, what will happen if instead of quarter wave plate we have a half wave plate and we launch a light with strata polarization oriented in such a way that $\varphi = \pi/4$. Now, in this case also we get the two components as E_y and E_z, but from equation number 17 and 18 you can see that this is not a circularly polarized wave, it is a linearly polarized wave with a direction of polarization making an angle of 135 degree with the z axis, you said that as you vary the thickness of the crystal when $\theta = \pi$, the output is no more circularly polarized light it is linearly polarized light, but the direction of vibration is now rotated. Initially the vibration was along $\varphi = \pi/4$, but now it is at 135 degree with the z axis.

Now, if we pass this beam through a calcite quarter wave plate then the emergent wave will again be a circularly polarized and in particularly we will get left circularly polarized light. Now, instead of launching a linearly polarized light on a half wave plate if you launch a right circularly polarized light normally on calcite half wave plate, the emergent beam will be left circularly polarized.

Thus, the half wave plate will have a thickness which would be given by equation number 19, this again came from equation number 11, but note that if the thickness d is such that θwhich is the phase difference is not equal to $\pi/2, \pi$ or $3\pi/2$ or 2π then the emergent beam will be elliptically polarized.

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For a positive crystal (like quartz), $n_e > n_0$, and

Eqn. (9) and (10) should be written as

$$E_y = E_0 \sin\phi \cos(\omega t + \theta') \qquad (20)$$

$$E_z = E_0 \cos\phi \cos\omega t \qquad (21)$$

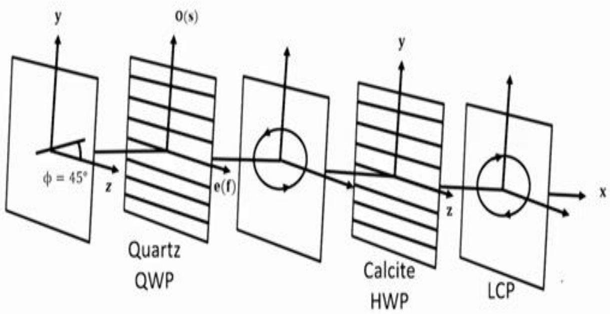
where $\theta' = \frac{\omega}{c} d(n_e - n_0)$

For a quarter wave plate,

$$d = (2m+1)\frac{\lambda_0}{4(n_e - n_0)} \qquad (22)$$

$$m = 0,1,2, \ldots\ldots\ldots$$

Hence if in the fig. (1), the calcite crystal quarter wave plate is replaced by a quartz QWP, the emergent beam will be left circularly polarized



By appropriately choosing the instant $t = 0$, the components may be written as

$$E_y = E_0 \sin\phi \cos(\omega t - \theta) \qquad (9)$$

$$E_z = E_0 \cos\phi \cos\omega t \qquad (10)$$

where,

$$\theta = (\theta_0 - \theta_e) = kd(n_0 - n_e) = \frac{\omega}{c}(n_0 - n_e)d \quad — \quad (11)$$

Eqn. (11) represents the phase difference between the ordinary and extra-ordinary beams.

Till now we just talked about negative cluster where n_o is greater than n_e, but what would be the situation if the crystal is positive, in positive crystal n_e would be larger than n_o. Now, if n_e is larger than n_o then previous equation 9 and 10 where we wrote the field components in terms of relative phase it will be changed.

Now, you see now let us go to equation number 9 and 10 you see here here the phase part is $\omega t - \theta$ but if n_e is larger than n_o then equation number 11 changes θ is negative now because the $n_o - n_e$ would be replaced by $n_e - n_o$, there would be one extra negative sign before θ and therefore equation 9 modifies and this modification results equation number 20 and here you see its $\omega t + \theta'$ where $\theta' = (\omega/c)d(n_e - n_0)$, and for quarter wave plate we know that $d = (2m + 1)\lambda_0/4(n_0 - n_e)$.

Now, observe the difference in the denominator instead of $n_o - n_e$ we now have $n_e - n_o$ this is the only difference and m is integer here. Hence in the first figure the calcite crystal quarter wave plate if it is replaced by a quartz quarter wave plate, the emergent wave beam will be left circularly polarized, which is shown here. This is all for today, I end my lecture with this and thank you for joining me.